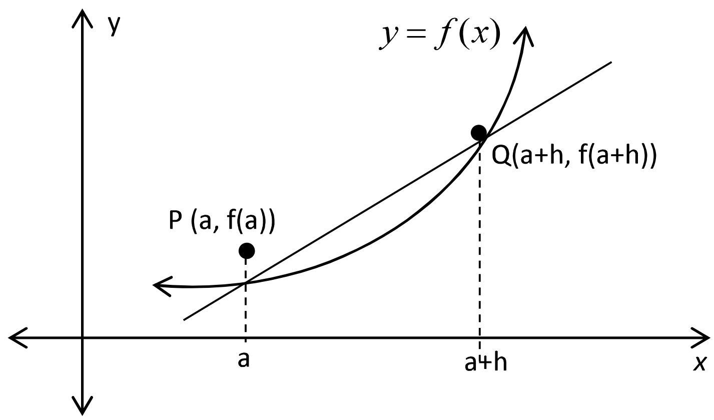
Rates of Change and the Slope of a Curve

The **average rate of change** (ARoC) of a function $f(x)$ between two points $P(a, f(a))$ and $Q(a+h, f(a+h))$ refers to the slope of the secant line, m_S, joining P and Q:



$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

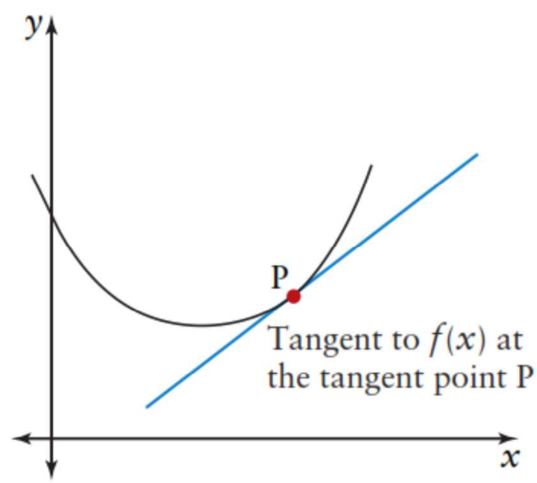
$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

$$\boxed{m_S = \frac{f(a+h) - f(a)}{h}, h \neq 0}$$

This expression is called the "Difference Quotient"

As Q gets closer and closer to P, the slope of the secant line can be used to approximate the slope of the line tangent to f at P, which we called the **instantaneous rate of change** (IRoC) of $f(x)$ at P.

The **instantaneous rate of change** (IRoC) of function $f(x)$ at point $P(a, f(a))$ on the function can be determined by the slopes of the secant lines PQ as point $Q(a+h, f(a+h))$ gets closer and closer to point P $(a, f(a))$, ie, as h gets closer and closer to zero, $a+h$ gets closer and closer to a, thus Q gets closer and closer to P and the slope of the secant line PQ gets closer and closer to the slope of the tangent line, m_T.



$$\boxed{m_S = \frac{f(a+h) - f(a)}{h} \rightarrow m_T \quad as \quad h \rightarrow 0}$$

Rates of Change and the Slope of a Curve

Average rate of change refers to the rate of change of a function over an interval. It corresponds to the slope of the **secant** connecting the two end points of the interval.

Instantaneous rate of change refers to the rate of change at a specific point. It corresponds to the slope of the tangent passing through a single point, or **tangent** point, on the graph of a function.

For a given function $y = f(x)$, the instantaneous rate of change at $x = a$ is **estimated** by calculating the slope of a secant over a very small interval, $a \le x \le a + h$, where h is a very small number (i.e. approaching zero). The slope of the secant between $P(a, f(a))$ and $Q(a+h, f(a+h))$ is:

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}, \quad h \neq 0}$$

This expression is called the **difference quotient**.

Example: A decorative balloon is being filled with helium. The table shows the volume of helium in the balloon at 3 second intervals for 30 seconds.

t (s)	0	3	6	9	12	15	18	21	24	27	30
V (cm^3)	0	4.2	33.5	113.0	267.9	523.3	904.3	1436.0	2143.6	3052.1	4186.7

a) Calculate the slope of the secant for each interval. What does the slope of the secant represent?

(i) 21s to 30s

$$\frac{\Delta y}{\Delta x} = \frac{4186.7 - 1436.0}{30 - 21} \doteq 306 \text{ cm}^3/\text{s}$$

(ii) 21 s to 27 s

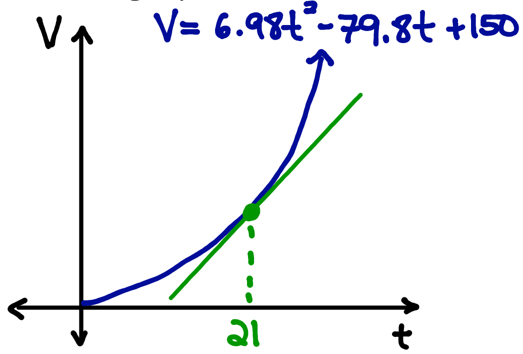
$$\frac{\Delta y}{\Delta x} = \frac{305.21 - 1436.0}{27 - 21} \doteq 269 \text{ cm}^3/\text{s}$$

(iii) 21 s to 24 s

$$\frac{\Delta y}{\Delta x} = \frac{2143.6 - 1436.0}{24 - 21} \doteq 236 \text{ cm}^3/\text{s}$$

The slope of the secant represents the average rate of change. In this case, it represents the rate at which the volume of the balloon is changing over an interval of time. (Note: a positive value represents an increase in the volume.)

b) Graph the information in the table using the GDC. Draw an approximate tangent at the point on the graph corresponding to 21 s and calculate the slope of this line. What does this graph illustrate? What does the slope of the tangent represent?



Tangent line using the GDC: $y = 213x - 2930$.
$\therefore$ The slope of the line is 213.

The graph represents the volume of the balloon increasing over time. The slope of the tangent represents the instantaneous rate of change of the volume of the balloon at exactly 21 s.

c) Compare the secant slopes in a) to the slope of the tangent. What do you notice? What information would you need to calculate a secant slope that is even closer to the slope of the tangent?

The slopes of the secants in part a) are 306 cm^3/s, 269 cm^3/s and 236 cm^3/s. These values are approaching the value of the slope of the tangent. To calculate a secant slope that is even closer to the slope of the tangent, data for a smaller interval of time should be used.

DETERMINING TANGENTS USING THE GDC

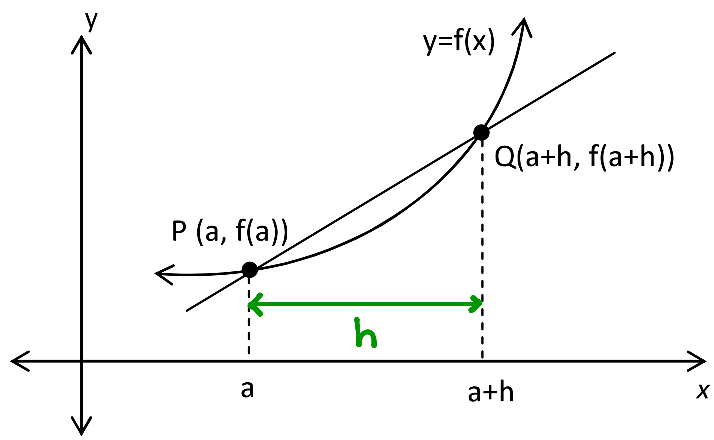
A graphing calculator can be used to draw a tangent to a curve when the equation for the function is known.

Steps:

1. Enter the equation: Y=
2. Press $\boxed{2^{nd}}$ and $\boxed{\text{PRGM}}$ to access the DRAW menu
3. Choose 5:Tangent(
4. Enter the x-value of the tangent point. [Press ENTER.]

Note: Equation of the tangent is at the bottom left of the screen at the given x-value.

Rates of Change Using Equations



For a given function $y = f(x)$, the instantaneous rate of change at $x = a$ is estimated by calculating the slope of a secant over a very small interval, $a \le x \le a + h$, where h is a very small number.

Slope of secant (average rate of change):

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}, \ h \neq 0$$

This expression is called the "Difference Quotient"

Example 1: Write a difference quotient that can be used to obtain an algebraic expression for estimating the slope of the tangent to a function $f(x) = x^3$ at x=2.

$$\frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{h}$$

$$= \frac{(2+h)^3 - (2)^3}{h}$$

$$= \frac{(2+h)^3 - 8}{h}$$

or

$f(x) = x^3 \rightarrow f(2) = 2^3 = 8$, $f(2+h) = (2+h)^3$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{(2+h)^3 - 8}{h}$$

Example 2: Ali is cleaning the outside of the balcony windows at his apartment which is located 90 m above the ground. He accidentally kicks a flowerpot, sending it over the edge of the balcony. The height of the flowerpot above the ground at any instant after it begins to fall is $s(t) = 90 - 4.9t^2$, where t is the time in seconds and s is the height in metres.

a) Determine the average rate of change of the flowerpot's height above the ground in the interval between 1 s and 3 s after it fell from the edge of the balcony.
b) Estimate the instantaneous rate of change of the flowerpot's height at 1 s and 3 s.
c) Determine the equation of the tangent at $t = 1$ by hand.
d) Verify your results in part c) using the GDC.

Solution:

a) $$\frac{\Delta s}{\Delta t} = \frac{s(3)-s(1)}{3-1}$$
$$= \frac{[90-4.9(3)^2]-[90-4.9(1)^2]}{2}$$
$$= -19.6$$
$\therefore$ The average rate of change is -19.6 m/s.

b) Difference Quotient

$$\frac{\Delta s}{\Delta t} = \frac{s(a+h)-s(a)}{h}$$
$$= \frac{[90-4.9(a+h)^2]-[90-4.9a^2]}{h}$$
$$= \frac{[90-4.9(a^2+2ah+h^2)]-90+4.9a^2}{h}$$
$$= \frac{90-4.9a^2-9.8ah-4.9h^2-90+4.9a^2}{h}$$
$$= \frac{-9.8ah-4.9h^2}{h}$$
$$= \frac{h(-9.8a-4.9h)}{h}$$
$$= -9.8a-4.9h, \ h \neq 0$$

Instantaneous Rate of Change:

At 1s, using $h=0.001$:
$$\frac{\Delta s}{\Delta t} = -9.8(1)-4.9(0.001)$$
$$\doteq -9.80$$
$\therefore$ The instantaneous rate of change at 1s is approx. -9.80 m/s.

At 3s, using $h=0.001$:
$$\frac{\Delta s}{\Delta t} = -9.8(3)-4.9(0.001)$$
$$= -29.4$$
$\therefore$ The IRoC is approx. -29.4 m/s.

c) From part b) the IRoC at 1s is approx -9.80 m/s.

$$s(1) = 90 - 4.9(1)^2$$
$$= 85.1$$

$\therefore$ Tangent point is $(1, 85.1)$.

Equation of tangent line:

$x = 1$
$y = 85.1$
$m = -9.80$

$$y = mx + b$$
$$(85.1) = (-9.80)(1) + b$$
$$94.9 = b$$

or

$$y - y_1 = m(x - x_1)$$
$$s - (85.1) = (-9.80)[t - (1)]$$
$$s = -9.80t + 9.80 + 85.1$$
$$s = -9.80t + 94.9$$

$\therefore$ The equation of the tangent line at $t = 1$ is $s = -9.80t + 94.9$

d) From the GDC, equation of the tangent is: $y = -9.8x + 94.9$

Experiencing Limits

Evaluating Limits of Convergent Sequences

A **sequence** is a function whose domain is the set of positive integers n=1,2,3,....

The values or individual terms of a sequence are generally denoted by a subscript of n on t. In other words, we use t_n rather than f(n).
For example, the list of all positive odd numbers forms the sequence 1,3,5,7,....
This sequence could be represented algebraically by two different formulas:

1. Recursive Formula

$t_n = t_{n-1} + 2 \; where \; t_1 = 1$

$t_2 = t_1 + 2 = 1 + 2 = 3$ $\quad t_3 = t_2 + 2 = 3 + 2 = 5$

2. Arithmetic Formula

$t_n = 1 + 2(n-1)$

$= 2n - 1 \quad for \;\; n = 1,2,3,...$

$t_1 = 2(1) - 1 = 1$
$t_2 = 2(2) - 1 = 3$
$t_3 = 2(3) - 1 = 5$

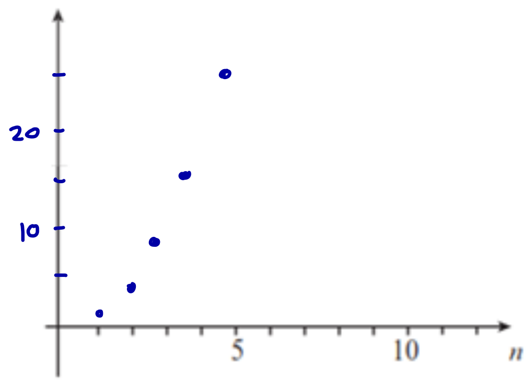
If we continue this sequence of numbers, would this sequence approach a single value?
In other words, as $n \to +\infty$ does t_n approach a limit?
As n increases, we see that t_n becomes arbitrarily large in value.
Therefore, as $n \to +\infty$, $t_n \to +\infty$.
We could use limits to write this as

$$\lim_{n \to \infty}(2n-1) = \infty$$

The behavior of infinite sequences
It is often very important to examine what happens to a sequence as n gets very large.
There are three types of behavior that we shall wish to describe explicitly. These are

- sequences that 'tend to infinity';
- sequences that '**converge** to a real limit';
- sequences that 'do not tend to a limit at all'.

First we look at sequences that tend to infinity. We say a sequence tends to infinity if, however large a number we choose, the sequence becomes greater than that number, and stays greater. Here are some examples of sequences that tend to infinity.

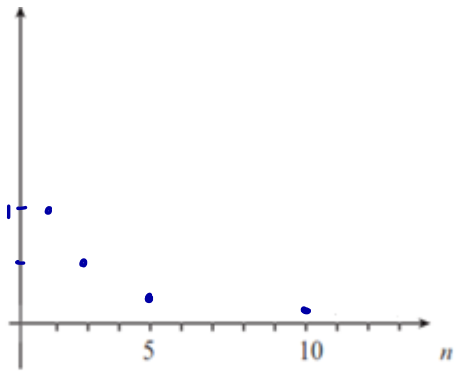


$a_n = n^2; n \geq 1$

n	$a_n = n^2$
1	1
2	4
5	25
10	100
50	2500

Now we look at sequences with real limits. We say a sequence tends to a real limit if there is a real number, **L**, such that the sequence gets closer and closer to it. We say **L** is the limit of the sequence.

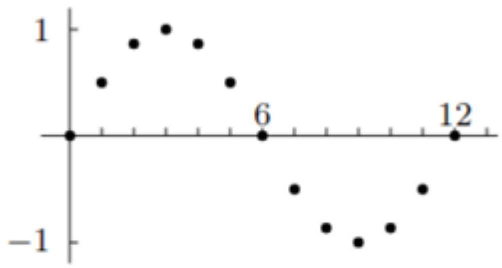
$$a_n = \frac{1}{n}; n \geq 1$$



n	$a_n = \frac{1}{n}$
1	1
2	0.5
5	0.2
10	0.1
15	0.06

And finally we look at sequences that cannot approach any specific number **L** as ***n*** grows large.

$$a_n = \sin\left(\frac{n\pi}{6}\right)$$



n	$a_n = \sin\left(\frac{n\pi}{6}\right)$
0	0
3	1
6	0
9	−1
12	0

Definitions:
We say that the sequence $\{a_n\}$ converges (or is convergent or has limit) if it tends to a number L.
A sequence diverges (or is divergent) if it does not tend to any number.

Exercises
Decide whether each of the following sequences tends to infinity , tends to a real limit, or does not tend to a limit at all. If a sequence tends to a real limit, work out what it is.

1. $a_n = 2^n$

As $n \to \infty$, $a_n \to \infty$

2. $a_n = 1000 - n$

As $n \to \infty$, $a_n \to -\infty$

3. $a_n = \frac{2n}{n+1}$

As $n \to \infty$, $a_n \to 2$

4. $a_n = \sin\left(\frac{n\pi}{2}\right)$

As $n \to \infty$, $a_n \to \{-1, 0, 1\}$

5. $a_n = \sqrt[2^n]{5}$

As $n \to \infty$. $a_n \to 1$

6. $a_n = 2 - \frac{1}{n^2}$

As $n \to \infty$, $a_n \to 2$

The limit of a Function

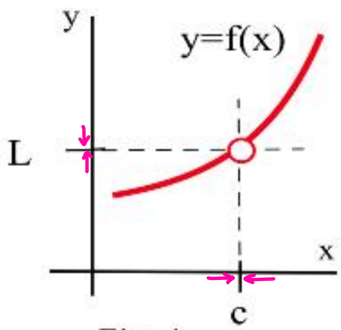


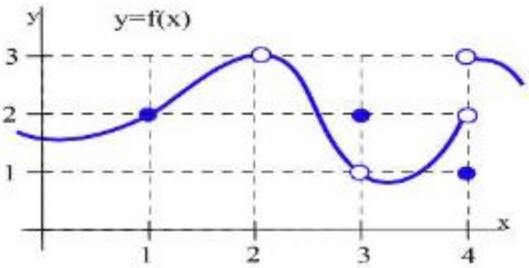
Fig. 1

Calculus has been called the study of continuous change, and the **limit** is the basic concept that describe and analyze such change. The limit of a function describes the behaviour of the function when the variable is near, **but does not equal**, a specific number (Fig.1).
If the values of ***f(x)*** get closer and closer, as close as we want, to one number **L** as we take values of x very close to (but not equal to) a number c ,then we say:

The limit of ***f(x)***, as x approaches **c** is **L** and we write: $\lim_{x \to c} f(x) = L$.

f(c) is the ONLY number that describes the behaviour (value) of f **AT** the point $x=c$.
$\lim_{x \to c} f(x)$ is a single number that describes the behaviour of f **NEAR, <u>BUT NOT AT</u>** point $x=c$. Note: $\lim_{x \to c} f(x)$ does not have to equal f(a) for the limit to exist.

<u>Example #1:</u>
Use the graph of y=f(x) and determine the following limits.



(a) $\lim_{x \to 2} f(x) =$ 3 Note: f(2) DNE

(b) $\lim_{x \to 3} f(x) =$ 1

(c) $\lim_{x \to 1} f(x) =$ 2

(d) $\lim_{x \to 4} f(x) =$ DNE Note Does not exist

One- Sided Limit

Sometimes, what happens to us at a place depends on the direction we use to approach that place. Similarly, the values of a function near a point may depend on the direction we use to approach that point. On the number line we can approach a point from the left or right, and that leads to **one-sided limits.**

The left limit as x approaches c of f(x) is L if the values of f(x) get as close to L as we want when x is very close to the left of c, x< c : $\lim_{x \to c^-} f(x) = L$

The right limit, written with $\lim_{x \to c^+} f(x)$ requires that x lie to the right of c, x>c.

One-sided Limit Theorem:

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

Corollary:
If $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$, then $\lim_{x\to c} f(x)$ *does not exist.*

Example #4:
Use the graph below to evaluate each of the following limits, if it exists.

a) $\lim_{x\to 1^+} f(x) =$ 2

b) $\lim_{x\to 1^-} f(x) =$ 2

c) $\lim_{x\to 1} f(x) =$ 2

d) $\lim_{x\to 2^+} f(x) =$ 1

e) $\lim_{x\to 2^-} f(x) =$ 1

f) $\lim_{x\to 2} f(x) =$ 1

g) $\lim_{x\to 3^+} f(x) =$ 2

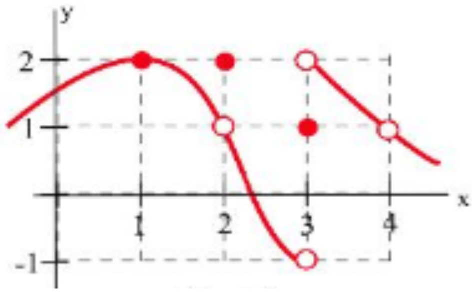
h) $\lim_{x\to 3^-} f(x) =$ -1

i) $\lim_{x\to 3} f(x) =$ DNE

j) $\lim_{x\to 4^+} f(x)$ 1

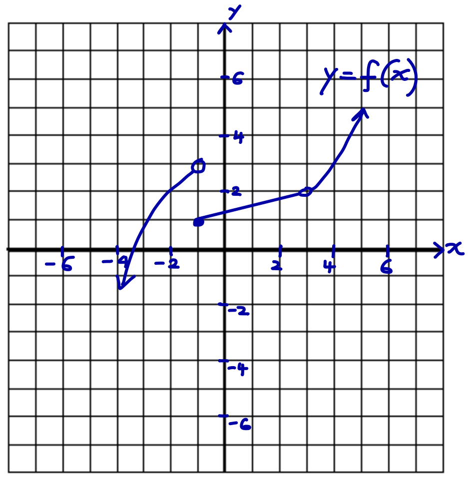
k) $\lim_{x\to 4^-} f(x) =$ 1

l) $\lim_{x\to 4} f(x) =$ 1



Example #5: Sketch the graph of a function that has the following characteristics:

- $\lim_{x\to -1^-} f(x) = 3$
- $\lim_{x\to -1^+} f(x) = 1$
- $\lim_{x\to 3} f(x) = 2$
- $f(3)$ is undefined



Example # 6: Consider the following piecewise function $f(x)$, where ***A*** and ***B*** are constants. $f(x)=\begin{cases} Ax+B & \text{if } x<-2 \\ x^2+2Ax-B & \text{if } -2\le x<1 \\ 4 & \text{if } x>1 \end{cases}$. Determine all values of the constants ***A*** and ***B*** so that $\lim_{x\to -2} f(x)$ and $\lim_{x\to 1} f(x)$ both exist.

- $\lim_{x\to -2} f(x)$ exists

$\lim_{x\to -2^-} Ax+B = \lim_{x\to -2^+} x^2+2Ax-B$

$-2A+B = 4-4A-B$

$2A+2B=4$

$A+B=2$ —— ①

- $\lim_{x\to 1} f(x)$ exists

$\lim_{x\to 1^-} x^2+2Ax-B = \lim_{x\to 1^+} 4$

$1+2A-B=4$

$2A-B=3$ —— ②

① + ②

$3A=5$

$\boxed{A=\frac{5}{3}}$

Sub in ①

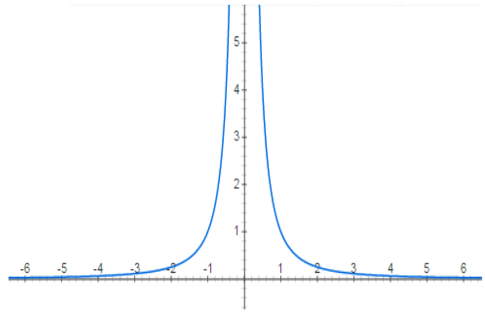
$\frac{5}{3}+B=2$

$B=2-\frac{5}{3}$

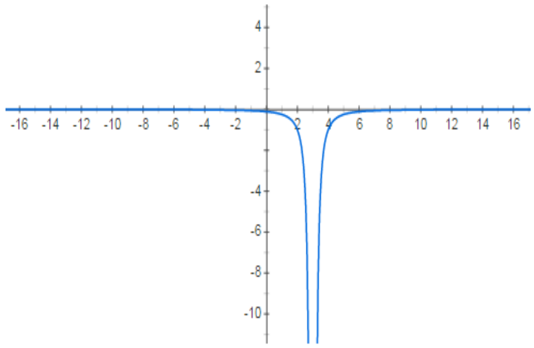
$\boxed{B=\frac{1}{3}}$

The value of A is $\frac{5}{3}$ and B is $\frac{1}{3}$

Infinite Limits

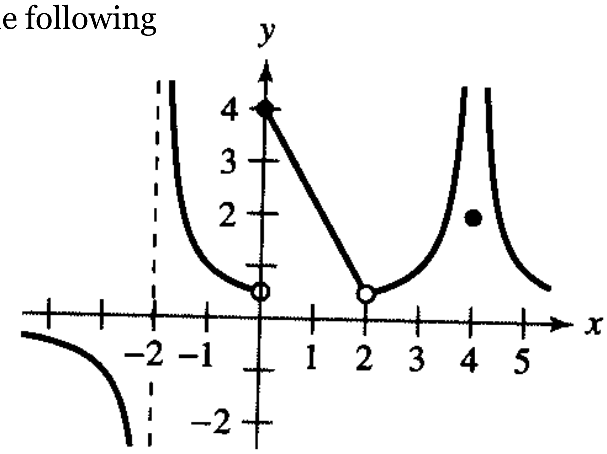


We say that $\lim\limits_{x \to a} f(x) = \infty$ if we can make f(x) arbitrarily large for all x sufficiently close to x=a, from both sides, without actually letting x= a .



We say that $\lim\limits_{x \to a} f(x) = -\infty$ if we can make f(x) arbitrarily large and negative for all x sufficiently close to x=a, from both sides, without actually letting x= a .

<u>EXAMPLE# 1</u>: Use the graph of f(x)below to find the following



a) $f(4) = 2$

b) $f(-2) =$ undefined

c) $\lim\limits_{x \to -2^+} f(x) = +\infty$

d) $\lim\limits_{x \to -2} f(x) =$ DNE

e) $\lim\limits_{x \to 2^+} f(x) = 0\ 5$

f) $\lim\limits_{x \to 4^-} f(x) = +\infty$

We can determine a limit intuitively, but we can also use properties of limits to evaluate limits.

Properties of limits

For any real numbers a ,c and k , suppose $f(x)$ and $g(x)$ both have limits at $x = a$.

1. $\lim\limits_{x \to a} k = k$
2. $\lim\limits_{x \to a} x = a$
3. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
4. $\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} f(x)$
5. $\lim\limits_{x \to a} [f(x) g(x)] = \left[\lim\limits_{x \to a} f(x)\right]\left[\lim\limits_{x \to a} g(x)\right]$
6. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, $\lim\limits_{x \to a} g(x) \neq 0$
7. $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$

Example #2 If $\lim_{x\to 0}\frac{f(x)}{x}=1$ and $\lim_{x\to 0}\frac{g(x)}{x}=2$, determine $\lim_{x\to 0}\frac{f(x)}{g(x)}$.

$$\frac{\lim_{x\to 0}\frac{f(x)}{x}}{\lim_{x\to 0}\frac{g(x)}{x}}=\frac{1}{2}$$

$$\frac{\frac{\lim_{x\to 0}f(x)}{\lim_{x\to 0}x}}{\frac{\lim_{x\to 0}g(x)}{\lim_{x\to 0}x}}=\frac{1}{2}$$

$$\frac{\lim_{x\to 0}f(x)}{\lim_{x\to 0}g(x)}=\frac{1}{2}$$

$$\therefore \boxed{\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{1}{2}}$$

Example #3 If $\lim_{x\to -3}f(x)=4$, use properties of limit to determine $\lim_{x\to -3}\frac{x\sqrt{f(x)}}{x^2+f(x)}$.

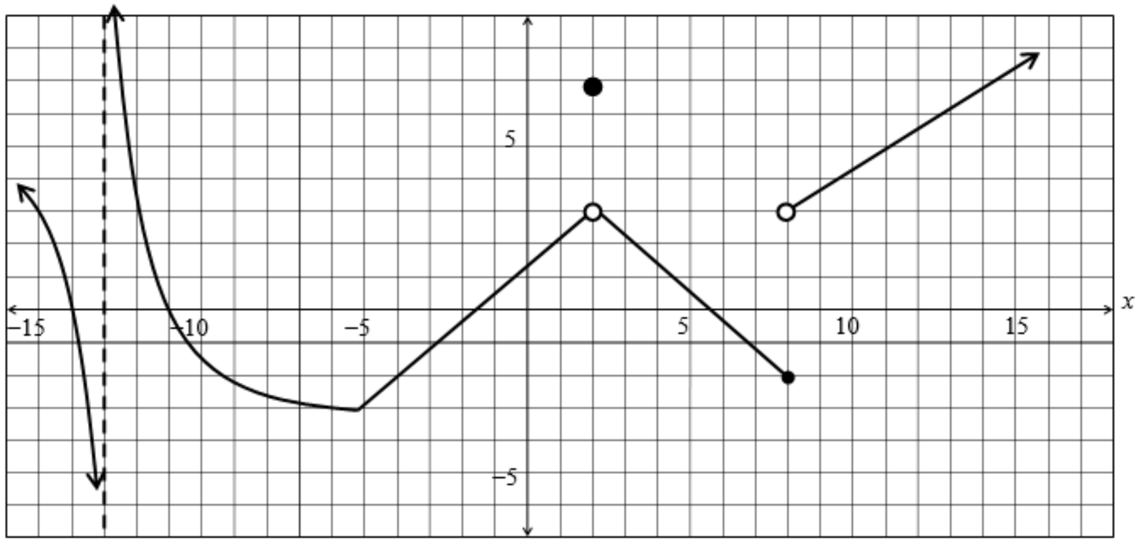
$$=\frac{\left[\lim_{x\to -3}x\right]\left[\lim_{x\to -3}f(x)\right]^{\frac{1}{2}}}{\lim_{x\to -3}x^2+\lim_{x\to -3}f(x)}$$

$$=\frac{(-3)(4)^{\frac{1}{2}}}{(-3)^2+4}$$

$$=\frac{-6}{13}$$

Practice

1. Consider the following graph of the function and evaluate the following limits.



a) $\lim\limits_{x\to 2} f(x) =$ _____ **b)** $f(2) =$ ____ **c)** $f(-5) =$ ____

d) $\lim\limits_{x\to 8^-} f(x) =$ _____ **e)** $\lim\limits_{x\to 8^+} f(x) =$ _____ **f)**

$\lim\limits_{x\to 8} f(x) =$ _____

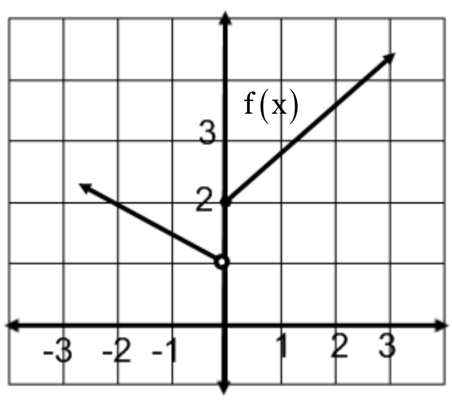
g) $\lim\limits_{x\to -13^-} f(x) =$ _____ **h)** $\lim\limits_{x\to -13^+} f(x) =$ _____ **i)** $f(-13) =$ ____

2. Consider $f(x)=\begin{cases} 7-x^2, & \text{if } x\le -2 \\ ax+b, & \text{if } -2<x<3 \\ \dfrac{3}{x}, & \text{if } x\ge 3 \end{cases}$. Determine values for *a* and *b* so that $\lim\limits_{x\to -2} f(x)$ and $\lim\limits_{x\to 3} f(x)$ exist.

3. Given $f(x)=\begin{cases} \dfrac{3x^2-5x-2}{x-2} & 0\le x<2 \\ x^2 & x<0 \\ \dfrac{14\left(\sqrt{x^2+12}-4\right)}{x-2} & 2<x \end{cases}$, find. $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 2} f(x)$.

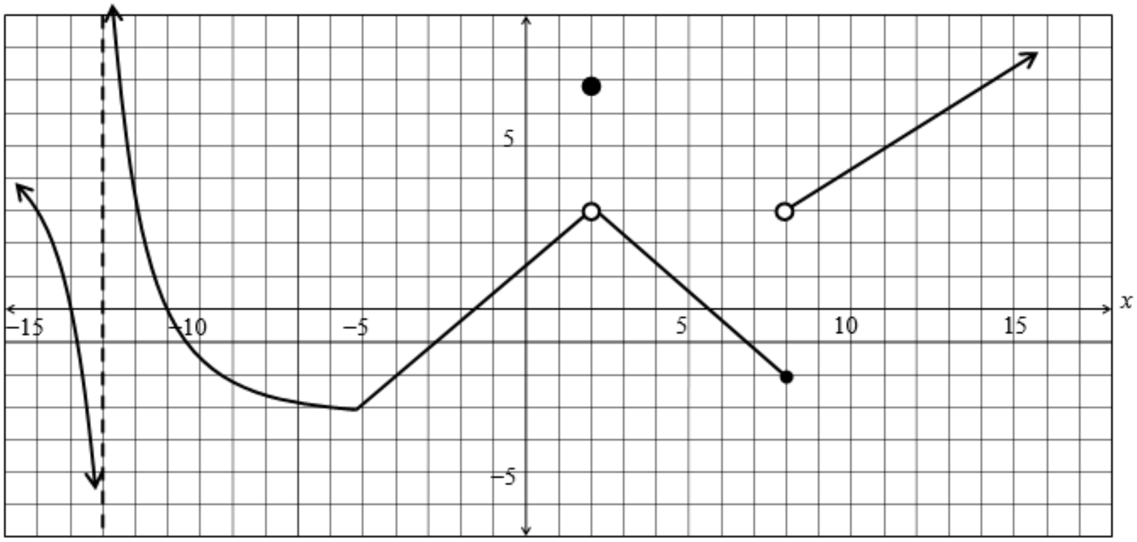
4. Let $g(x) = Ax + B$, where A and B are constants. $\lim\limits_{x\to 1} g(x)=-2$ and $\lim\limits_{x\to -1} g(x)=4$, find the values of A and B.

5. The graph of function f(x) is shown. Find the $\lim\limits_{x\to 1}\left(f(x-1)+f(1-x)\right)$.



Practice ——Solution

1. Consider the following graph of the function and evaluate the following limits.



a) $\lim_{x\to 2} f(x) = \mathbf{3}$ **b)** $f(2) = \mathbf{7}$ **c)** $f(-5) = \mathbf{-3}$

d) $\lim_{x\to 8^-} f(x) = \mathbf{-2}$ **e)** $\lim_{x\to 8^+} f(x) = \mathbf{3}$ **f)** $\lim_{x\to 8} f(x) =$ **DNE**

g) $\lim_{x\to -13^-} f(x) = -\infty$ **h)** $\lim_{x\to -13^+} f(x) = \infty$ **i)** $f(-13) =$ **undefined**

2. Consider $f(x) = \begin{cases} 7 - x^2, & \text{if } x \le -2 \\ ax + b, & \text{if } -2 < x < 3 \\ \frac{3}{x}, & \text{if } x \ge 3 \end{cases}$. Determine values for *a* and *b* so that $\lim_{x\to -2} f(x)$ and $\lim_{x\to 3} f(x)$ exist.

$\lim_{x\to -2} f(x)$ to exist : $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x)$

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \left(7 - x^2\right) = 3$$
$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (ax + b) = -2a + b$$
$$\Longrightarrow \quad -2a + b = 3$$

$\lim_{x\to 3} f(x)$ to exist : $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax + b) = 3a + b$$
$$\Longrightarrow \quad 3a + b = 1$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \left(\frac{3}{x}\right) = 1$$

Solving the linear system of equations

$$\begin{cases} -2a + b = 3 \\ 3a + b = 1 \end{cases} \Rightarrow \boxed{a = \frac{-2}{5}} \,\&\, \boxed{b = \frac{11}{5}}$$

3. Given $f(x)=\begin{cases} \dfrac{3x^2-5x-2}{x-2} & 0\le x<2 \\ x^2 & x<0 \\ \dfrac{14\left(\sqrt{x^2+12}-4\right)}{x-2} & 2<x \end{cases}$, find. $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 2} f(x)$.

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\left(x^2\right)=0$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{3x^2-5x-2}{x-2}=1$$

$\because \lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$ $\therefore \lim\limits_{x\to 0} f(x)$ DNE

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\frac{3x^2-5x-2}{x-2}$$

$$=\lim_{x\to 2^-}\frac{(3x+1)\cancel{(x-2)}}{\cancel{x-2}}$$

$$=7$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\frac{14\left(\sqrt{x^2+12}-4\right)}{x-2}$$

$$=\lim_{x\to 2^+}\frac{14\left(\sqrt{x^2+12}-4\right)}{x-2}\times\frac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4}$$

$$=\lim_{x\to 2^+}\frac{14\left(x^2+12-16\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2^+}\frac{14\left(x^2-4\right)}{(x-2)\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2^+}\frac{14(x+2)\cancel{(x-2)}}{\cancel{(x-2)}\left(\sqrt{x^2+12}+4\right)}$$

$$=\lim_{x\to 2^+}\frac{14(x+2)}{\sqrt{x^2+12}+4}$$

$$=\frac{56}{8}$$

$$=7$$

$\because \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x) = 7$

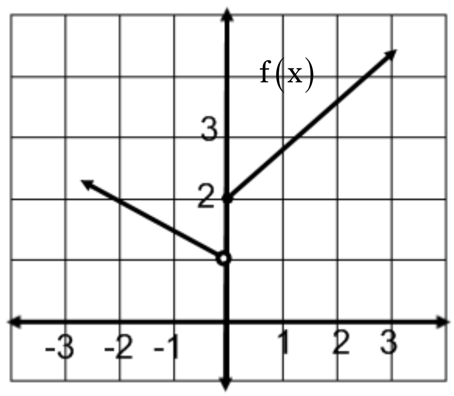
$\therefore \lim\limits_{x\to 2} f(x) = 7$

4. Let $g(x) = Ax + B$, where A and B are constants. $\lim\limits_{x\to 1} g(x) = -2$ and $\lim\limits_{x\to -1} g(x) = 4$, find the values of A and B.

$$\lim_{x\to 1} g(x) = -2 \Rightarrow \lim_{x\to 1}(Ax+B) = -2$$
$$A + B = -2$$
$$\lim_{x\to -1} g(x) = - \Rightarrow \lim_{x\to -1}(Ax+B) = 4$$
$$-A + B = 4$$

$$\begin{cases} A+B=-2 \\ -A+B=4 \end{cases} \quad \boxed{A=-3}\ ,\boxed{B=1}$$
$$\therefore g(x) = -3x+1$$

5. The graph of function f(x) is shown. Find the $\lim\limits_{x\to 1}\left(f(x-1)+f(1-x)\right)$.



$$\lim_{x\to 1^-}\left(f(x-1)+f(1-x)\right) = \lim_{x\to 1^-} f(x-1) + \lim_{x\to 1^-} f(1-x)$$
$$= f(0^-) + f(0^+)$$
$$= 1 + 2$$
$$= 3$$
$$\lim_{x\to 1^+}\left(f(x-1)+f(1-x)\right) = \lim_{x\to 1^+} f(x-1) + \lim_{x\to 1^+} f(1-x)$$
$$= f(0^+) + f(0^-)$$
$$= 2 + 1$$
$$= 3$$
$$\therefore \lim_{x\to 1}\left(f(x-1)+f(1-x)\right) = 3$$

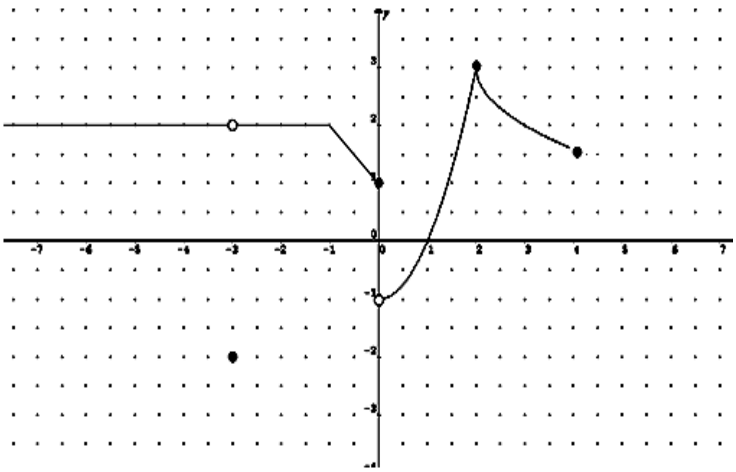
Warm up

1. Use the graph of the piece-wise function $y = f(x)$ shown below to answer the following questions.

a) $\lim_{x \to 2^+} f(x) = $ 3

b) $\lim_{x \to 0} f(x) = $ DNE $\lim_{x \to 0^-} f(x) = 1$ $\lim_{x \to 0^+} f(x) = -1$

c) $f(-3) = $ -2



2. Find all values of a and b such that for the function $f(x) = \begin{cases} ax - 2 & x < -1 \\ x^2 - bx + a & -1 \le x < 3 \\ 4 & x \ge 3 \end{cases}$

$\lim_{x \to 3} f(x)$ and $\lim_{x \to -1} f(x)$ both exist.

$\lim_{x \to 3} f(x)$ exists

$\lim_{x \to 3^-} x^2 - bx + a = \lim_{x \to 3^+} 4$

$9 - 3b + a = 4$

$a - 3b = -5$ — 1

$\lim_{x \to -1} f(x)$ exists

$\lim_{x \to -1^-} ax - 2 = \lim_{x \to -1^+} x^2 - bx + a$

$-a - 2 = 1 + b + a$

$b = -2a - 3$ — 2

Sub 2 in 1

$a - 3(-2a - 3) = -5$

$a + 6a + 9 = -5$

$7a = -14$

$\boxed{a = -2}$

Sub in 1

$-2 - 3b = -5$

$-3b = -3$

$\boxed{b = 1}$

Evaluating limit

To evaluate a limit algebraically, we can use the following methods:

- **direct substitution**
- **factoring**
- **rationalizing**
- **one-sided limits**
- **change of variable**

METHOD 1: Direct Substitution

Example: Evaluate the following limits.

a) $\lim_{x\to 2}\left(x^2-4x+1\right)$

$= (2)^2-4(2)+1$

$= 4-8+1$

$= -3$

b) $\lim_{x\to 3}\frac{x-2}{x+2}$

$= \frac{(3)-2}{(3)+2}$

$= \frac{1}{5}$

c) $\lim_{x\to \frac{\pi}{2}}\left(\sin x-\cos 2x\right)$

$= \sin\left(\frac{\pi}{2}\right)-\cos 2\left(\frac{\pi}{2}\right)$

$= 1-(-1)$

$= 2$

d) $\lim_{x\to 8}\frac{\log_x 512+\sqrt{1+x}}{x^2+x^{\frac{1}{3}}-8}$

$= \frac{\log_{(8)} 512+\sqrt{1+8}}{(8)^2+(8)^{\frac{1}{3}}-8}$

$= \frac{3+3}{64+2-8}$

$= \frac{6}{58}$

$= \frac{3}{29}$

e) $\lim_{x\to 2}\frac{3x^2}{x-2}$

$= \frac{3(2)^2}{(2)-2}$

$= \frac{12}{0}$

$=$ DNE

Note $\lim_{x\to 2^-}\frac{3x^2}{x-2}$

$= \frac{3(2)^2}{0^-}$

$= -\infty$

$\lim_{x\to 2^+}\frac{3x^2}{x-2}$

$= \frac{12}{0^+}$

$= +\infty$

METHOD 2: Factoring

This method is used on questions where direct substitution yields $\frac{0}{0}$, which is referred to as an '**indeterminate form**.' Whenever this happens, simplify the expression by factoring and reducing, expanding and simplifying, or by finding a common denominator.

Example#4: Evaluate the following limits.

a) $\lim_{x\to 3}\frac{x-3}{x^2-4x+3}$

$= \lim_{x\to 3}\frac{x-3}{(x-3)(x-1)}$

$= \lim_{x\to 3}\frac{1}{x-1}$

$= \frac{1}{(3)-1}$

$= \frac{1}{2}$

b) $\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

$= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

$= \lim_{x\to 1}\frac{x^2+x+1}{x+1}$

$= \frac{(1)^2+(1)+1}{(1)+1}$

$= \frac{3}{2}$

c) $\lim_{x\to 1}\frac{x^2+5x-6}{x^3-x^2-4x+4}$

$= \lim_{x\to 1}\frac{(x+6)(x-1)}{x^2(x-1)-4(x-1)}$

$= \lim_{x\to 1}\frac{(x+6)(x-1)}{(x-1)(x^2-4)}$

$= \lim_{x\to 1}\frac{x+6}{x^2-4}$

$= \frac{(1)+6}{(1)^2-4}$

$= -\frac{7}{3}$

d) $\lim_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$

$= \lim_{x\to 2}\dfrac{\frac{2-x}{2x}}{x-2}$

$= \lim_{x\to 2}\dfrac{2-x}{2x}\cdot\dfrac{1}{-(2-x)}$

$= \lim_{x\to 2}\dfrac{-1}{2x}$

$= \dfrac{-1}{2(2)}$

$= \dfrac{-1}{4}$

e) $\lim_{h\to 0}\dfrac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h}$

$= \lim_{h\to 0}\dfrac{4-(2+h)^2}{4(2+h)^2}\cdot\dfrac{1}{h}$

$= \lim_{h\to 0}\dfrac{4-4-4h-h^2}{4h(2+h)^2}$

$= \lim_{h\to 0}\dfrac{-h(4+h)}{4h(2+h)^2}$

$= \lim_{h\to 0}\dfrac{-(4+h)}{4(2+h)^2}$

$= \dfrac{-4}{4(4)}$

$= -\dfrac{1}{4}$

METHOD 3: Rationalizing

When the expression we are trying to find the limit of is a fraction involving a square root, it sometimes works to rationalize. Multiply by its Conjugate

a) $\lim_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$

$= \lim_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}\cdot\dfrac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$

$= \lim_{x\to 0}\dfrac{1+x-1}{x(\sqrt{1+x}+1)}$

$= \lim_{x\to 0}\dfrac{1}{\sqrt{1+x}+1}$

$= \dfrac{1}{2}$

b) $\lim_{t\to -2}\dfrac{3t^2+4t-4}{\sqrt{6+t}-\sqrt{2-t}}$

$= \lim_{t\to -2}\dfrac{3t^2+4t-4}{\sqrt{6+t}-\sqrt{2-t}}\cdot\dfrac{\sqrt{6+t}+\sqrt{2-t}}{\sqrt{6+t}+\sqrt{2-t}}$

$= \lim_{t\to -2}\dfrac{(3t-2)(t+2)(\sqrt{6+t}+\sqrt{2-t})}{6+t-2+t}$

$= \lim_{t\to -2}\dfrac{(3t-2)(t+2)(\sqrt{6+t}+\sqrt{2-t})}{2(2+t)}$

$= \lim_{t\to -2}\dfrac{(3t-2)(\sqrt{6+t}+\sqrt{2-t})}{2}$

$= \dfrac{[3(-2)-2][\sqrt{6+(-2)}+\sqrt{2-(-2)}]}{2}$ $= \dfrac{-8(4)}{2}$

$= -16$

c) $\lim_{x\to 3}\dfrac{3-x}{\sqrt{13+x}-\sqrt{7+x^2}}\cdot\dfrac{\sqrt{13+x}+\sqrt{7+x^2}}{\sqrt{13+x}+\sqrt{7+x^2}}$

$= \lim_{x\to 3}\dfrac{(3-x)(\sqrt{13+x}+\sqrt{7+x^2})}{13+x-7-x^2}$

$= \lim_{x\to 3}\dfrac{(3-x)(\sqrt{13+x}+\sqrt{7+x^2})}{-(x^2-x-6)}$

$= \lim_{x\to 3}\dfrac{-(x-3)(\sqrt{13+x}+\sqrt{7+x^2})}{-(x-3)(x+2)}$

$= \lim_{x\to 3}\dfrac{\sqrt{13+x}+\sqrt{7+x^2}}{x+2}$

$= \dfrac{\sqrt{13+(3)}+\sqrt{7+(3)^2}}{(3)+2}$ $= \dfrac{4+4}{5}$

$= \dfrac{8}{5}$

d) $\lim_{x\to 2}\dfrac{\sqrt{x+2}-2}{\sqrt{7+x}-3}\cdot\dfrac{\sqrt{x+2}+2}{\sqrt{x+2}+2}\cdot\dfrac{\sqrt{7+x}+3}{\sqrt{7+x}+3}$

$= \lim_{x\to 2}\dfrac{(x+2-4)(\sqrt{7+x}+3)}{(7+x-9)(\sqrt{x+2}+2)}$

$= \lim_{x\to 2}\dfrac{(x-2)(\sqrt{7+x}+3)}{(x-2)(\sqrt{x+2}+2)}$

$= \lim_{x\to 2}\dfrac{\sqrt{7+x}+3}{\sqrt{x+2}+2}$

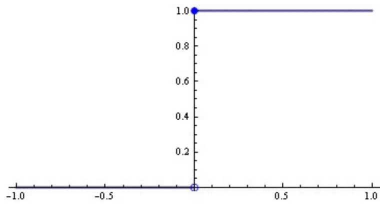
$= \dfrac{\sqrt{7+(2)}+3}{\sqrt{(2)+2}+2}$

$= \dfrac{6}{4}$

$= \dfrac{3}{2}$

METHOD 4: One-sided Limits

a) Consider the Heaviside[1] function $H(t)=\begin{cases}0 & , \ t<0 \\ 1 & , \ t\geq 0\end{cases}$



Evaluate:

i) $\lim\limits_{t\to 0^-} H(t) = 0$ ii) $\lim\limits_{t\to 0^+} H(t) = 1$ iii) $\lim\limits_{t\to 0} H(t) = \text{DNE}$

b) $\lim\limits_{x\to 2}\dfrac{x^2+|x-2|-4}{|x-2|}$

$\lim\limits_{x\to 2^+}\dfrac{x^2+x-2-4}{x-2}$

$=\lim\limits_{x\to 2^+}\dfrac{(x+3)(x-2)}{x-2}$

$=\lim\limits_{x\to 2^+} x+3$

$=(2)+3$

$=5$

$\lim\limits_{x\to 2^-}\dfrac{x^2-(x-2)-4}{-(x-2)}$

$=\lim\limits_{x\to 2^-}\dfrac{x^2-x-2}{-(x-2)}$

$=\lim\limits_{x\to 2^-}\dfrac{(x-2)(x+1)}{-(x-2)}$

$=\lim\limits_{x\to 2^-} -(x+1)$

$=-[(2)+1]$

$=-3$

$\lim\limits_{x\to 2}\dfrac{x^2+|x-2|-4}{|x-2|}$

$=\text{DNE}$

METHOD 5: Change of Variable

Evaluate the following limits, if they exist. Show your work.

a) $\lim\limits_{x\to 0}\dfrac{x}{\sqrt[3]{x+1}-1}$

Let $(x+1)^{\frac{1}{3}}=u \Leftarrow$ As $x\to 0$, $u\to 1$

$x+1=u^3$

$x=u^3-1$

$=\lim\limits_{u\to 1}\dfrac{u^3-1}{u-1}$

$=\lim\limits_{u\to 1}\dfrac{(u-1)(u^2+u+1)}{u-1}$

$=\lim\limits_{u\to 1} u^2+u+1$

$=(1)^2+(1)+1$

$=3$

b) $\lim\limits_{x\to 1}\dfrac{2x+3\sqrt{x}-5}{\sqrt{x}-1}$

Let $x^{\frac{1}{2}}=u$

$x=u^2$

As $x\to 1$, $u\to 1$

$=\lim\limits_{u\to 1}\dfrac{2u^2+3u-5}{u-1}$

$=\lim\limits_{u\to 1}\dfrac{(2u+5)(u-1)}{u-1}$

$=\lim\limits_{u\to 1} 2u+5$

$=2(1)+5$

$=7$

---

[1] This function is named after the electrical engineer Oliver Heaviside (1850-1925) and can be used to describe an electric current that is turned on at time t = 0.

c) $\lim_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{x}$

Let $(x+8)^{\frac{1}{3}}=u$
$x+8=u^3$
$x=u^3-8$
As $x\to 0$, $u\to 2$

$=\lim_{u\to 2}\frac{u-2}{u^3-8}$

$=\lim_{u\to 2}\frac{u-2}{(u-2)(u^2+2u+4)}$

$=\lim_{u\to 2}\frac{1}{u^2+2u+4}$

$=\frac{1}{(2)^2+2(2)+4}$

$=\frac{1}{12}$

d) $\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$

Let $u=x^{\frac{1}{6}}$
$u^2=x^{\frac{1}{3}}$
$u^3=x^{\frac{1}{2}}$
As $x\to 1$, $u\to 1$

$=\lim_{u\to 1}\frac{u^3-1}{u^2-1}$

$=\lim_{u\to 1}\frac{(u-1)(u^2+u+1)}{(u-1)(u+1)}$

$=\lim_{u\to 1}\frac{u^2+u+1}{u+1}$

$=\frac{(1)^2+(1)+1}{(1)+1}$

$=\frac{3}{2}$

Practice

1. If $\lim_{x\to -1}\frac{x+1}{x^3-ax^2-x+6}$ exists, find the value of a.
2. If $\lim_{x\to 2}\frac{x-7}{x^2+ax+b}=-\infty$,find the value of a+b.
3. Give an example of functions $f(x)$ and $g(x)$ such that $\lim_{x\to 0}\left(f(x)+g(x)\right)$ exists but $\lim_{x\to 0}f(x)$ and $\lim_{x\to 2}f(x)$ do not exist.
4. Give an example of a function such that $\lim_{x\to 0}[f(x)]^2$ exists but $\lim_{x\to 0}f(x)$ does not exist.
5. Evaluate

   (a) $\lim_{x\to 56}\frac{\sqrt[3]{x+8}-4}{\sqrt{x-40}-4}$ (b) $\lim_{x\to \frac{1}{4}}\frac{4x-1}{\frac{1}{\sqrt{x}}-2}$ (c) $\lim_{x\to 0}\frac{(1+2x)^{\frac{1}{6}}-1}{x}$

6. Is there a value of k for which $\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$ exists? If so, find k and find the values of the limit.

Practice —Solution

1. If $\lim\limits_{x\to-1} \dfrac{x+1}{x^3-ax^2-x+6}$ exists, find the value of a.

Let $f(x)=x^3-ax^2-x+6$. If $f(-1)=0$ then:

$0=(-1)^3-a(-1)^2-(-1)+6$

$0=-1-a+1+6$

$0=-a+6 \Rightarrow \boxed{a=6}$

If $a=6$ then:

$\lim\limits_{x\to-1}\dfrac{x+1}{x^3-6x^2-x+6}$

$=\lim\limits_{x\to-1}\dfrac{x+1}{x^2(x-6)-(x-6)}$

$=\lim\limits_{x\to-1}\dfrac{x+1}{(x-6)(x^2-1)}$

$\lim\limits_{x\to-1}\dfrac{\cancel{x+1}}{(x-6)(x-1)\cancel{(x+1)}}=\dfrac{1}{14}$

If $a\neq 6$ then $\lim\limits_{x\to-1}\dfrac{x+1}{x^3-ax^2-x+6}=0$

In either case the limit exists, therefore $a\in\mathbb{R}$.

* For $\lim\limits_{x\to-1}\dfrac{f(x)}{g(x)}$ to exist, $x+1$ must be a factor of $f(x)$ and $g(x)$ (since this will be a removable discontinuity) $\therefore f(-1)=0$

2. If $\lim\limits_{x\to2}\dfrac{x-7}{x^2+ax+b}=-\infty$,find the value of a+b.

If $\lim\limits_{x\to2}\dfrac{x-7}{x^2+ax+b}=-\infty$, since $\lim\limits_{x\to2}(x-7)=-5$ then $(x-2)^2$ must be a factor of x^2+ax+b.

(Recall: $\dfrac{-5}{0^+}=-\infty$)

Let $f(x)=x^2+ax+b$ We now know that $f(x)=(x-2)^2$.

Therefore:

$x^2+ax+b=x^2-4x+4$

$a=-4, b=4$ by observation

$\therefore a+b=0$

Denominator must → 0 with direct substitution so that the limit is $-\infty$

$\therefore$ The only way to have the denominator 0 when subbing in $x=2$ is if the factors are $(x-2)^2$

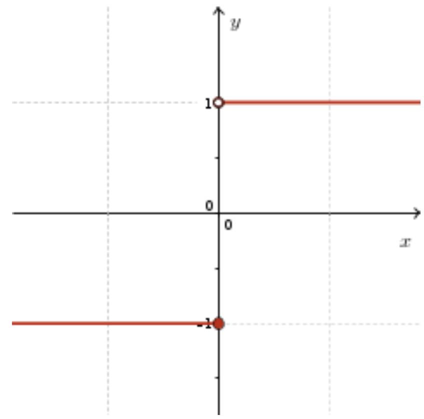
3. Give an example of functions $f(x)$ and $g(x)$ such that $\lim_{x\to 0}(f(x)+g(x))$ exists but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

Answers may vary; two such functions are $f(x)=\dfrac{1}{x}$ *and* $g(x)=\dfrac{-1}{x}$.

4. Give an example of a function such that $\lim_{x\to 0}[f(x)]^2$ exists but $\lim_{x\to 0} f(x)$ does not exist.

There are many possible solutions to this question.

One possible function is $f(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x\le 0\end{cases}$



$\lim_{x\to 0}[f(x)]^2=1$. However $\lim_{x\to 0^-} f(x)=-1$

so $\lim_{x\to 0} f(x)$ *does not exist.*

5. Evaluate

(a) $\lim_{x\to 56}\dfrac{\sqrt[3]{x+8}-4}{\sqrt{x-40}-4}$

Let: $u^3=x+8$

$u^3-8=x$ As $x\to 56, u\to 4$

$$\lim_{x\to 56}\frac{\sqrt[3]{x+8}-4}{\sqrt{x-40}-4}=\lim_{u\to 4}\frac{u-4}{\sqrt{u^3-8-40}-4}$$

$$=\lim_{u\to 4}\frac{u-4}{\sqrt{u^3-48}-4}\times\frac{\sqrt{u^3-48}+4}{\sqrt{u^3-48}+4}$$

$$=\lim_{u\to 4}\frac{(u-4)\left(\sqrt{u^3-48}+4\right)}{u^3-48-16}$$

$$=\lim_{u\to 4}\frac{(u-4)\left(\sqrt{u^3-48}+4\right)}{u^3-64}$$

$$=\lim_{u\to 4}\frac{\cancel{(u-4)}\left(\sqrt{u^3-48}+4\right)}{\cancel{(u-4)}\left(u^2+4u+16\right)}$$

$$=\lim_{u\to 4}\frac{\left(\sqrt{u^3-48}+4\right)}{\left(u^2+4u+16\right)}$$

$$=\frac{8}{48}=\frac{1}{6}$$

(b) $\lim_{x\to\frac{1}{4}}\dfrac{4x-1}{\dfrac{1}{\sqrt{x}}-2}$

Let: $u^2=x$ $\to u=\sqrt{x}$, As $x\to\frac{1}{4}, u\to\frac{1}{2}$

$$\lim_{x\to\frac{1}{4}}\frac{4x-1}{\frac{1}{\sqrt{x}}-2}=\lim_{u\to\frac{1}{2}}\frac{4u^2-1}{\frac{1}{u}-2}$$

$$=\lim_{u\to\frac{1}{2}}\frac{4u^2-1}{\frac{1-2u}{u}}$$

$$=\lim_{u\to\frac{1}{2}}\frac{\cancel{(2u-1)}(2u+1)u}{-\cancel{(2u-1)}}$$

$$=-1$$

(c) $\lim_{x\to 0}\frac{(1+2x)^{\frac{1}{6}}-1}{x}$

$1+2x=u^6$

$x=\frac{u^6-1}{2}$ As $x\to 0,\ u\to 1$

$$\lim_{x\to 0}\frac{(1+2x)^{\frac{1}{6}}-1}{x}=2\lim_{u\to 1}\frac{u-1}{u^6-1}$$

$$=2\lim_{u\to 1}\frac{\cancel{u-1}}{\cancel{(u-1)}\left(u^5+u^4+u^3+u^2+u+1\right)}$$

$$=\frac{2}{6}$$

$$=\frac{1}{3}$$

6. Is there a value of *k* for which $\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$ exists? If so, find *k* and find the values of the limit.

If $\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$ exists, then we must have:

$k(-2)^2-6(-2)+3-k=0$ and $(-2)^2+3(-2)+2=0$

$4k+12+3-k=0$

$3k=-15$

$\boxed{k=-5}$

* For $\lim_{x\to -2}\frac{f(x)}{g(x)}$ to exist, $x+2$ must be a factor of $f(x)$ and $g(x)$ (since this will be a removable discontinuity) $\therefore f(-2)=0$

$$\lim_{x\to -2}\frac{-5x^2-6x+8}{x^2+3x+2}$$

$$=\lim_{x\to -2}\frac{-(x+2)(5x-4)}{(x+2)(x+1)}$$

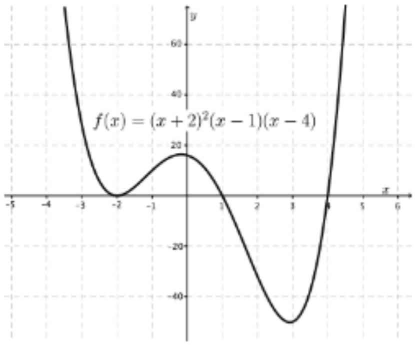
$$=\lim_{x\to -2}\frac{-(5x-4)}{(x+1)}$$

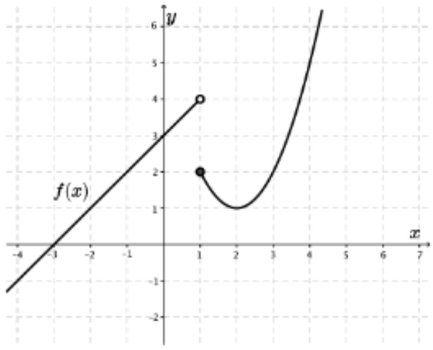
$$=-14$$

Limits & Continuity

A function is **continuous** if you can draw its graph **without lifting** your pencil. If the curve has holes or gaps, it is discontinuous, or has a discontinuity, at the point at which the gap occurs.

continuous vs. discontinuous



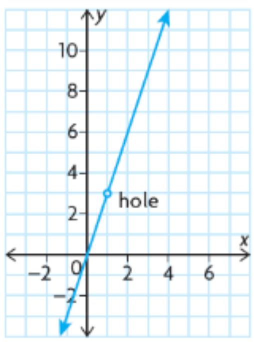


A function that is not continuous at x=a is referred to as **discontinuous** at **a**. The point, **a**, is known as a point of discontinuity.

Types of Discontinuities

There are four different types of discontinuities that we will discuss

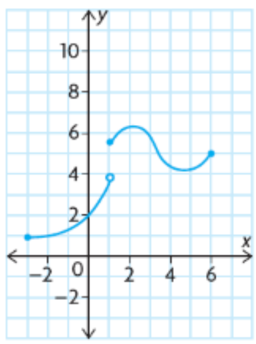
(1) Removable Discontinuity



$\lim_{x\to a} f(x) \neq f(a)$

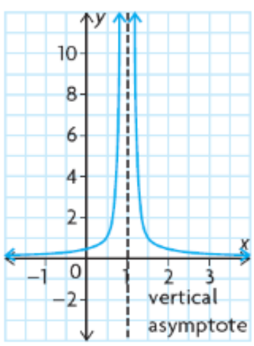
or $f(a)$ DNE

(2) Jump Discontinuity



$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$

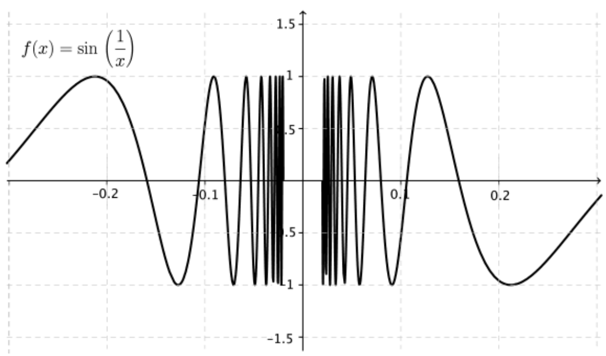
(3) Infinite Discontinuity



$\lim_{x\to a} f(x) = \infty$

or $\lim_{x\to a} f(x) = -\infty$

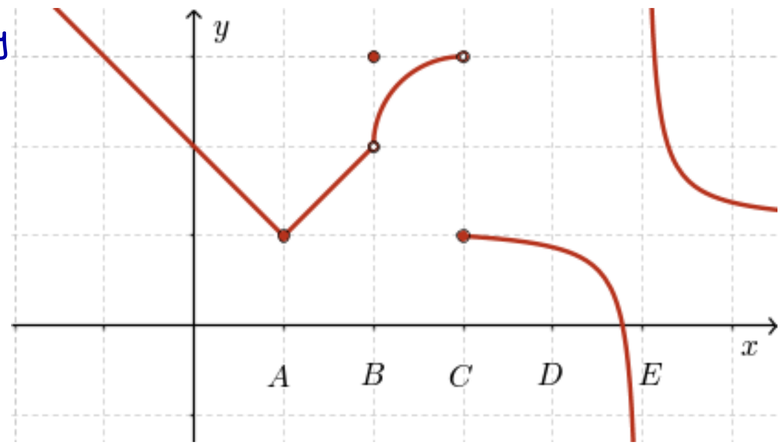
(4) Oscillating Discontinuities



$f(0)$ is not defined

Example#1: The graph of y=f(x) is shown. Determine whether the function is continuous at the indicated points. State the type of discontinuity (removable, jump, infinite, or none of these).

a. x=A Continuous at $x=A$
b. x=B Removable discontinuity
c. x=C Jump discontinuity
d. x=D Continuous at $x=D$
e. x=E Infinite discontinuity



Limit definition of Continuity

A function $f(x)$ is continuous at a value $x = a$ if the following **three conditions** are satisfied:

1. $f(a)$ **must exists**
2. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$
3. $\lim_{x\to a} f(x) = f(a)$

A function, $f(x)$, is continuous at $x=a$ if
$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Example#2: $f(x)=\begin{cases} \dfrac{x^2-6x}{x} & if \quad x\neq 0 \\ 2k-1 & if \quad x=0 \end{cases}$ and f is continuous at x = 0. Find the value of k.

If $f(x)$ is continuous at $x=0$, then $\lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0} \frac{x^2-6x}{x}$

$= \lim_{x\to 0} \frac{x(x-6)}{x}$

$= \lim_{x\to 0} x-6$

$= -6$

sub

$f(0) = 2k-1$

$-6 = 2k-1$

$-5 = 2k$

$\boxed{k = -\frac{5}{2}}$

Example#3: Find values of ***a*** and ***b*** that makes function f(x) continuous on $\mathbb{R}$.

$$f(x)=\begin{cases} ax+2b & \text{if } x\le -1 \\ x^2+2 & \text{if } -1<x\le 2 \\ 2ax-4b & \text{if } x>2 \end{cases}$$

If $f(x)$ is continuous on R, then $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$ and $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\lim_{x\to -1^-} ax+2b = \lim_{x\to -1^+} x^2+2$

$a(-1)+2b = (-1)^2+2$

$-a+2b=3$ — ①

$\lim_{x\to 2^-} x^2+2 = \lim_{x\to 2^+} 2ax-4b$

$2^2+2 = 2a(2)-4b$

$6 = 4a-4b$

$3 = 2a-2b$ — ②

① + ② $\boxed{6=a}$

sub in ①

$-6+2b=3$

$2b=9$

$\boxed{b=\frac{9}{2}}$

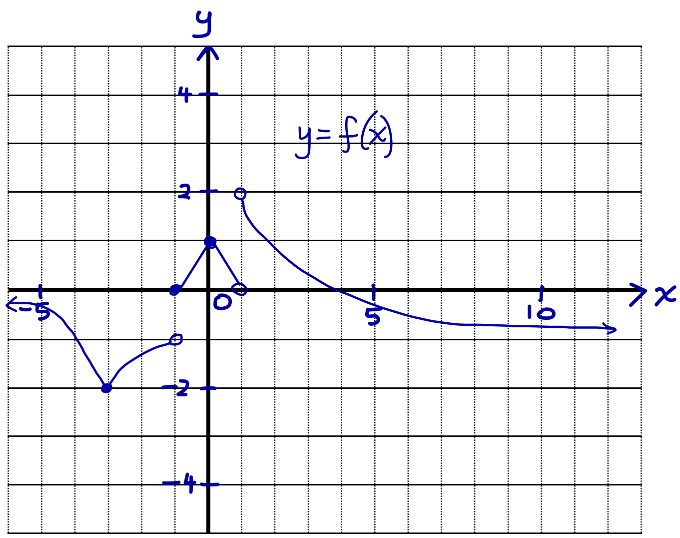
Example#4: Sketch the graph of a function that satisfies **all** of the following conditions:

continuous → $\lim_{x\to -3} f(x) = -2$, jump or infinite $\lim_{x\to -1} f(x)$ DNE, jump discontinuity $\lim_{x\to 1^-} f(x) = 0$, $\lim_{x\to 1^+} f(x) = 2$, $\lim_{x\to\infty} f(x) = -1$, ← HA at $y=-1$

$f(0)=1$, and f(x) is continuous over the interval $(-\infty,-1)$.

↑ y-intercept

★ Answers will vary



Practice:

Ex) Determine if $h(x) = \begin{cases} x-1, & x<-3 \\ x, & x=-3 \\ x^2-13, & -3<x\le 0 \\ 13-x, & x>0 \end{cases}$ is continuous over all real numbers. If a discontinuity exists, state where it is and the type.

$\lim_{x\to -3^-} x-1$
$= -3-1$
$= -4$

$\lim_{x\to -3^+} x^2-13$
$= (-3)^2-13$
$= -4$

$\lim_{x\to -3} h(x) = -4$

$h(-3) = -3$

$\because \lim_{x\to -3} h(x) \neq h(-3)$

$\therefore$ Function is discontinuous at $x=-3$
Removable discontinuity

$\lim_{x\to 0^-} x^2-13$
$= 0-13$
$= -13$

$\lim_{x\to 0^+} 13-x$
$= 13-0$
$= 13$

$\therefore \lim_{x\to 0} h(x) = \text{DNE}$

$\therefore$ The function is discontinuous at $x=0$.
This is a jump discontinuity

Ex) Determine if $k(x) = \begin{cases} x^2, & x<1 \\ -2, & x=1 \\ -x+3, & 1<x<4 \\ \frac{1}{2}x-3, & x\ge 4 \end{cases}$ is continuous over all real numbers. If a discontinuity exists, state where it is and the type.

$\lim_{x\to 1^-} x^2$
$= 1$

$\lim_{x\to 1^+} -x+3$
$= -1+3$
$= 2$

$\lim_{x\to 1} k(x) = \text{DNE}$

$\therefore$ Function is discontinuous at $x=1$ and it is a jump discontinuity.

$\lim_{x\to 4^-} -x+3$
$= -4+3$
$= -1$

$\lim_{x\to 4^+} \frac{1}{2}x-3$
$= \frac{1}{2}(4)-3$
$= -1$

$\therefore \lim_{x\to 4} k(x) = -1$

$k(4) = \frac{1}{2}x-3$
$= \frac{1}{2}(4)-3$
$= -1$

$\because \lim_{x\to 4} k(x) = k(4)$

$\therefore$ The function is continuous at $x=4$

Ex) Find the values of a and b that make $f(x)$ continuous for $x\epsilon R$. $f(x) = \begin{cases} ax+2b, & x \le -1 \\ x^2+2, & -1 < x \le 2 \\ 2ax-4b, & x > 2 \end{cases}$

For the function to be continuous, the limit as x approaches a must exist.

$$\lim_{x\to -1^-} ax+2b = \lim_{x\to -1^+} x^2+2$$

$$-a+2b=3 \quad — ①$$

$$\lim_{x\to 2^-} x^2+2 = \lim_{x\to 2^+} 2ax-4b$$

$$6 = 4a-4b$$

$$3 = 2a-2b \quad — ②$$

① + ②:

$$\boxed{a=6}$$

Sub in ①:

$$-6+2b=3$$

$$2b=9$$

$$\boxed{b=\frac{9}{2}}$$

Ex) Find the values of a and b that make $f(x)$ continuous for $x\epsilon R$. $f(x) = \begin{cases} b\cdot e^x+a+1, & x \le 0 \\ ax^2+b(x+3), & 0 < x \le 1 \\ a\cos(\pi x)+7bx, & x > 1 \end{cases}$

$[a=3, b=2]$

$$\lim_{x\to 0^-} be^x+a+1 = b+a+1 \qquad \lim_{x\to 0^+} ax^2+b(x+3) = 3b$$

$\therefore$ If $f(x)$ is continuous at $x=0$,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

$$b+a+1 = 3b$$

$$a-2b=-1 \quad — ①$$

Sub ② in ①:

$$\frac{3b}{2}-2b=-1$$

$$-\frac{b}{2}=-1$$

$$\boxed{b=2}$$

$$\lim_{x\to 1^-} ax^2+b(x+3) = a+4b$$

$$\lim_{x\to 1^+} a\cos(\pi x)+7bx = -a+7b$$

$$\therefore a+4b=-a+7b$$

$$2a=3b$$

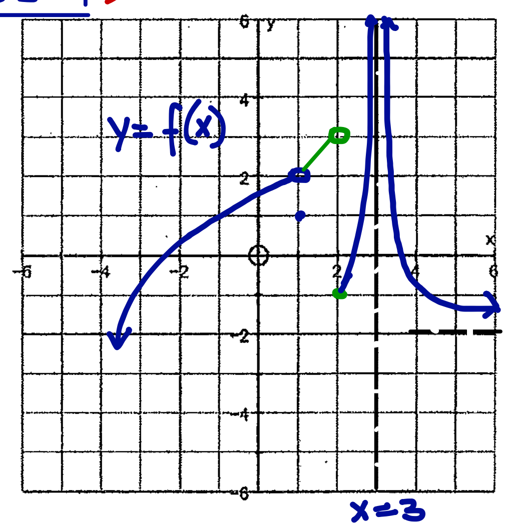
$$a=\frac{3b}{2} \quad — ②$$

Sub in ②:

$$\boxed{a=3}$$

Ex) Sketch the graph of a function f with the following properties:

- $\lim_{x\to 1} f(x) = 2$
- $f(1) = 1$
- $\lim_{x\to 2^+} f(x) = -1$
- $\lim_{x\to 2^-} f(x) = 3$
- $\lim_{x\to 3} f(x) = +\infty$
- $\lim_{x\to \infty} f(x) = -2$
- $\lim_{x\to -\infty} f(x) = -\infty$



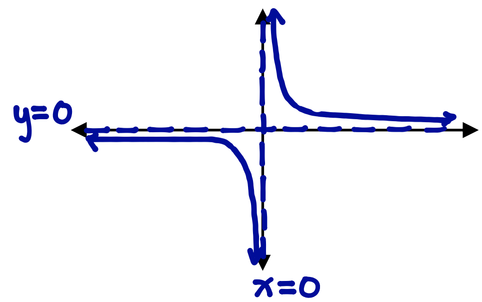
The Limit as x Approaches Infinity of a Rational Function

Recall: $f(x) = \frac{1}{x}$. $\lim_{x\to\infty} \frac{1}{x} = \underline{0}$ and $\lim_{x\to-\infty} \frac{1}{x} = \underline{0}$

Calculator Method:

$$\frac{1}{very\ large\ \#} = 0$$

Graphical Method:



As $x \to \pm\infty$, $f(x) \to 0$

Given a rational function f(x) $\lim_{x\to\infty} f(x) = \begin{cases} 0 \\ \pm\infty \\ \frac{a}{b} \end{cases}$, where $a, b \epsilon I, b \neq 0$

Determine the limit as x approaches infinity of $f(x) = \frac{P_m(x)}{Q_n(x)}$, where m and n are the degrees of the polynomials $P(x)$ and $Q(x)$, respectively.

Strategy: Divide each term by the term with the highest exponent. in the denominator.

Case 1: If m<n

Ex: $\lim_{x\to\infty} \frac{x+1}{x^2+4}$

$$= \lim_{x\to\infty} \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{4}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2}}{1 + \frac{4}{x^2}}$$

$$= 0$$

Therefore, if $m < n$, $\lim_{x\to\infty} f(x) = 0$

Case 2: If m>n

Ex: $\lim_{x\to\infty} \frac{x^3+x^2+4}{3x+1}$

$$= \lim_{x\to\infty} \frac{\frac{x^3}{x}+\frac{x^2}{x}+\frac{4}{x}}{\frac{3x}{x}+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{x^2+x+\frac{4}{x}}{3+\frac{1}{x}}$$

$$= \infty$$

What about $\lim_{x\to-\infty} \frac{x^2+x-3}{3x+1}$?

$$= \lim_{x\to-\infty} \frac{x+1-\frac{3}{x}}{3+\frac{1}{x}}$$

$$= -\infty$$

Therefore, if $m > n$, $\lim_{x\to\infty} f(x) = \pm\infty$

Case 3: If m=n

Ex: $\lim_{x\to\infty} \frac{x^3+2x+3}{4x^3+x^2+x}$

$$= \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{2x}{x^3}+\frac{3}{x^3}}{\frac{4x^3}{x^3}+\frac{x^2}{x^3}+\frac{x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{2}{x^2}+\frac{3}{x^3}}{4+\frac{1}{x}+\frac{1}{x^2}}$$

$$= \frac{1}{4}$$

Therefore, if $m = n$, $\lim_{x\to\infty} f(x) = \frac{a}{b}$, where $a,b \in I, b \neq 0$

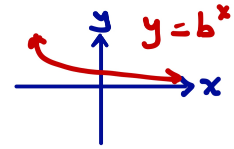
Practise: Evaluate the following.

a) $\lim\limits_{x\to\infty} \dfrac{x^2+6x^5}{x^3-7x^2+x^5}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^3}+6}{\frac{1}{x^2}-\frac{7}{x^3}+1}$

$= 6$

b) $\lim\limits_{x\to\infty} \dfrac{1}{3^x}$

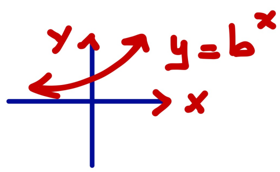


$= \lim\limits_{x\to\infty} \left(\frac{1}{3}\right)^x$

$= 0$

Note: $0<b<1$

c) $\lim\limits_{x\to\infty} \left(\dfrac{4}{3}\right)^x$



$= \infty$

Note: $b>1$

d) $\lim\limits_{x\to\infty} \dfrac{x+2x^2+5}{3x^2-6}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}+2+\frac{5}{x^2}}{3-\frac{6}{x^2}}$

$= \dfrac{2}{3}$

e) $\lim\limits_{x\to\infty} \dfrac{x^3+x^4+3x^6}{x^6+x^4+3x^3}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{x^3}+\frac{1}{x^2}+3}{1+\frac{1}{x^2}+\frac{3}{x^3}}$

$= 3$

f) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^4+5}}{x^2+6}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{x^4\left(1+\frac{5}{x^4}\right)}}{x^2+6}$

$= \lim\limits_{x\to\infty} \dfrac{x^2\sqrt{1+\frac{5}{x^4}}}{x^2\left(1+\frac{6}{x^2}\right)}$

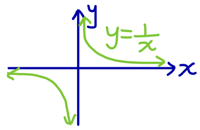
$= 1$

Limits at Infinity

When you are finding a limit at infinity, direct substitution can yield another indeterminate form $\frac{\infty}{\infty}$.To find the limit in this case, divide the functions in the numerator and denominator by the highest power of x in the denominator.

Example #5: Determine the following limits.

a) $\lim\limits_{x\to\infty}\frac{1}{x}$



$= 0$

b) $\lim\limits_{x\to\infty}\frac{x^2-4x+1}{x-4}$

$= \lim\limits_{x\to\infty}\dfrac{\frac{x^2}{x^2}-\frac{4x}{x^2}+\frac{1}{x^2}}{\frac{x}{x^2}-\frac{4}{x^2}}$

$= \lim\limits_{x\to\infty}\dfrac{1-\frac{4}{x}+\frac{1}{x^2}}{\frac{1}{x}-\frac{4}{x^2}} = +\infty$

c) $\lim\limits_{x\to\infty}\frac{5x^2-3x+4}{2x^2+x-7}$

$= \lim\limits_{x\to\infty}\dfrac{5-\frac{3}{x}+\frac{4}{x^2}}{2+\frac{1}{x}-\frac{7}{x^2}}$

$= \frac{5}{2}$

d) $\lim\limits_{x\to\infty}\left(\frac{1}{x}-\frac{1}{x+1}\right)$

$= \lim\limits_{x\to\infty}\dfrac{x+1-x}{x(x+1)}$

$= \lim\limits_{x\to\infty}\dfrac{1}{x^2+x}$

$= \lim\limits_{x\to\infty}\dfrac{\frac{1}{x^2}}{1+\frac{1}{x}}$

$= 0$

e) $\lim\limits_{x\to\infty}\frac{3x^5-2x^3+1}{(1-x)^5}$

$= \lim\limits_{x\to\infty}\dfrac{x^5\left(3-\frac{2}{x^2}+\frac{1}{x^5}\right)}{\left[x\left(\frac{1}{x}-1\right)\right]^5}$

$= \lim\limits_{x\to\infty}\dfrac{3x^5}{-x^5}$

$= -3$

Example #6: Find the limit of each function

(a) $\lim\limits_{x\to\infty}\frac{3}{x^4}$

$= 0$

(b) $\lim\limits_{x\to-\infty}\frac{2x^2+3}{3x^2-x+4}$

$= \lim\limits_{x\to-\infty}\dfrac{2+\frac{3}{x^2}}{3-\frac{1}{x}+\frac{4}{x^2}}$

$= \frac{2}{3}$

(c) $\lim\limits_{x\to\infty}\dfrac{1000x-3}{\frac{x}{1000}+1}$

$= \lim\limits_{x\to\infty}\dfrac{1000-\frac{3}{x}}{\frac{1}{1000}+\frac{1}{x}}$

$= 1000\,000$

(d) $\lim\limits_{x\to\infty}\frac{\sqrt{x}}{2\sqrt{x}+1}\cdot\frac{\sqrt{x}}{\sqrt{x}}$

$= \lim\limits_{x\to\infty}\dfrac{x}{2x+\sqrt{x}}$

$= \lim\limits_{x\to\infty}\dfrac{x}{x\left(2+\frac{1}{\sqrt{x}}\right)}$

$= \frac{1}{2}$

or d) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{2\sqrt{x}+1}$

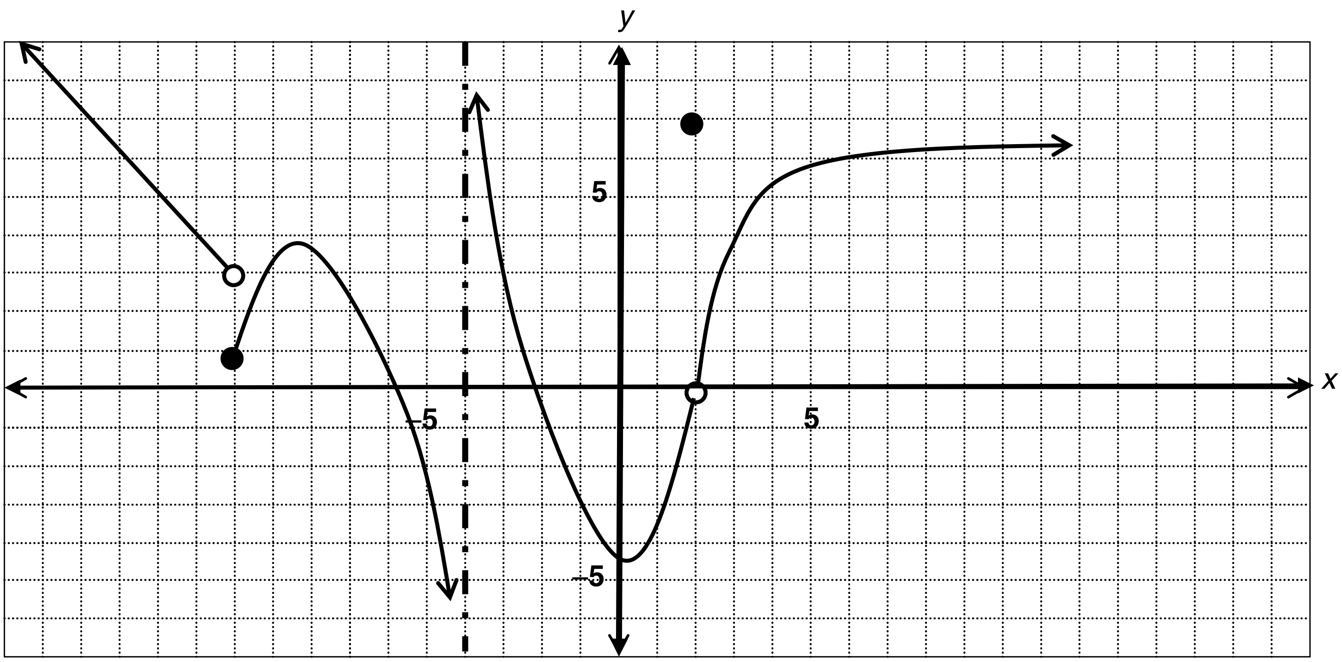
$= \lim\limits_{x\to\infty}\dfrac{1}{2+\frac{1}{\sqrt{x}}}$

$= \frac{1}{2}$

SOLUTION

MID-REVIEW
PUSHING YOUR BRAIN TO THE LIMIT

1. Consider the following graph of $f(x)$.



Evaluate the following limits. If the limit does not exist you must provide a reason.

i) $\lim_{x\to -10^+} f(x)$ $=1$

ii) $\lim_{x\to -10^-} f(x)$ $=3$

iii) $f(2)$ 7

iv) $\lim_{x\to 2^+} f(x)$ $=0$

v) $\lim_{x\to 2^-} f(x)$ $=0$

vi) $\lim_{x\to 2} f(x)$ $=0$

vii) $\lim_{x\to -4^+} f(x)$ $=\infty$

viii) $\lim_{x\to -4^-} f(x)$ $=-\infty$

ix) $\lim_{x\to \infty} f(x)$ 6

2. Based on graph above, answer the following questions:

a) An example of a **removable discontinuity** is when x=__**2**__

b) An example of a **jump discontinuity** is when x=_**-10**___

c) An example of an **infinite discontinuity** is when x=_**-4**___

SOLUTION

3. Evaluate the following indeterminate limits

a) $\lim\limits_{x\to\infty}\dfrac{4^{x-10}}{5^{x+10}}$

$$=\lim_{x\to\infty}\frac{4^x\times 4^{-10}}{5^x\times 5^{10}}$$

$$=\lim_{x\to\infty}\left(\frac{4}{5}\right)^x\times\frac{1}{20^{10}}$$

$$=0$$

b) $\lim\limits_{x\to -2}\dfrac{x^3+x^2-x+2}{x^2-x-6}$

$$=\lim_{x\to -2}\frac{\cancel{(x+2)}\left(x^2-x+1\right)}{\cancel{(x+2)}(x-3)}$$

$$=\lim_{x\to -2}\frac{x^2-x+1}{x-3}$$

$$=-\frac{7}{5}$$

c) $\lim\limits_{x\to 3^+}\dfrac{x+3}{x^2-9}$

$$=\lim_{x\to 3^+}\frac{\cancel{x+3}}{\cancel{(x+3)}(x-3)}$$

$$=\lim_{x\to 3^+}\frac{1}{(x-3)}$$

$$=\infty$$

d) $\lim\limits_{x\to 9}\dfrac{x\sqrt{x}-27}{\sqrt{x}-3}$

Let: $\sqrt{x}=u$

$x=u^2$

$$=\lim_{u\to 3}\frac{u^2u-27}{u-3}$$

$$=\lim_{u\to 3}\frac{u^3-27}{u-3}$$

$$=\lim_{u\to 3}\frac{\cancel{(u-3)}\left(u^2+3u+9\right)}{\cancel{u-3}}$$

$$=27$$

e) $\lim\limits_{x\to 5}\dfrac{\sqrt{x^2-9}-4}{\sqrt{x-1}-2}$

$$=\lim_{x\to 5}\frac{\sqrt{x^2-9}-4}{\sqrt{x-1}-2}\times\frac{\sqrt{x^2-9}+4}{\sqrt{x^2-9}+4}\times\frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$$

$$=\lim_{x\to 5}\frac{\left(x^2-9-16\right)\left(\sqrt{x-1}+2\right)}{(x-1-4)\left(\sqrt{x^2-9}+4\right)}$$

$$=\lim_{x\to 5}\frac{\left(x^2-25\right)\left(\sqrt{x-1}+2\right)}{(x-5)\left(\sqrt{x^2-9}+4\right)}$$

$$=\lim_{x\to 5}\frac{\cancel{(x-5)}(x+5)\left(\sqrt{x+1}+2\right)}{\cancel{(x-5)}\left(\sqrt{x^2-9}+4\right)}$$

$$=\lim_{x\to 5}\frac{(x+5)\left(\sqrt{x+1}+2\right)}{\left(\sqrt{x^2-9}+4\right)}$$

$$=\frac{[(5)+5]\left[\sqrt{(5)+1}+2\right]}{\left[\sqrt{(5)^2-9}+4\right]}$$

$$=\frac{40}{8}$$

$$=5$$

f) $\lim\limits_{x\to 10}\dfrac{(x-2)^{\frac{1}{3}}-2}{x-10}$

Let: $(x-2)^{\frac{1}{3}}=u$

$x-2=u^3\to x=u^3+2$

$$=\lim_{u\to 2}\frac{u-2}{u^3+2-10}$$

$$=\lim_{u\to 2}\frac{u-2}{u^3-8}$$

$$=\lim_{u\to 2}\frac{\cancel{u-2}}{\cancel{(u-2)}\left(u^2+2u+4\right)}$$

$$=\frac{1}{12}$$

g) $\lim\limits_{x\to 3}\dfrac{x^{-2}-3^{-2}}{x-3}$

$$=\lim_{x\to 3}\frac{\frac{1}{x^2}-\frac{1}{3^2}}{x-3}$$

$$=\lim_{x\to 3}\frac{9-x^2}{9x^2(x-3)}$$

$$=\lim_{x\to 3}\frac{-\cancel{(x-3)}(x+3)}{9x^2\cancel{(x-3)}}$$

$$=\lim_{x\to 3}\frac{-(x+3)}{9x^2}$$

$$=-\frac{2}{27}$$

h) $\lim\limits_{x\to\infty}\dfrac{5x^2-3x+4}{2x^2+x-7}$

$$=\lim_{x\to\infty}\frac{x^2\left(5-\cancel{\frac{3}{x}}^{0}+\cancel{\frac{4}{x^2}}^{0}\right)}{x^2\left(2+\cancel{\frac{1}{x}}^{0}-\cancel{\frac{7}{x^2}}^{0}\right)}$$

$$=\frac{5}{2}$$

i) $\lim\limits_{x\to\infty}\dfrac{\left(1-x^2\right)^3}{\left(3-2x^3\right)^2}$

$$=\lim_{x\to\infty}\frac{\left[x^2\left(\frac{1}{x^2}-1\right)\right]^3}{\left[x^3\left(\frac{3}{x^3}-2\right)\right]^2}$$

$$=\lim_{x\to\infty}\frac{x^6\left(\frac{1}{x^2}-1\right)^3}{x^6\left(\frac{3}{x^3}-2\right)^2}$$

$$=\lim_{x\to\infty}\frac{\left(\cancel{\frac{1}{x^2}}^{0}-1\right)^3}{\left(\cancel{\frac{3}{x^3}}^{0}-2\right)^2}$$

$$=\frac{(-1)^3}{(-2)^2}$$

$$=-\frac{1}{4}$$

SOLUTION

4. Is there a value of *k* for which $\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$ exists? If so, find *k* and find the values of the limit.

If $\lim_{x\to -2}\frac{kx^2-6x+3-k}{x^2+3x+2}$ exists, then we must have:

$k(-2)^2-6(-2)+3-k=0 \;\; and \;\; (-2)^2+3(-2)+2=0$

$4k+12+3-k=0$

$3k=-15$

$\boxed{k=-5}$

$$\lim_{x\to -2}\frac{-5x^2-6x+8}{x^2+3x+2}$$

$$=\lim_{x\to -2}\frac{-(x+2)(5x-4)}{(x+2)(x+1)}$$

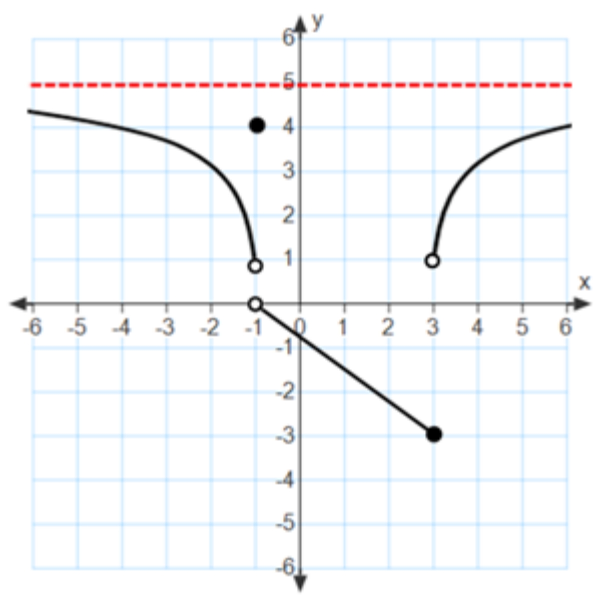
$$=\lim_{x\to -2}\frac{-(5x-4)}{(x+1)}$$

$$=-14$$

* For $\lim_{x\to -2}\frac{f(x)}{g(x)}$ to exist, $x+2$ must be a factor of $f(x)$ and $g(x)$ (since this will be a removable discontinuity) $\therefore f(-2)=0$

5. Create a sketch of a function f(x) where the following restrictions are met:
(There is not one correct answer. All that is required is that you meet these conditions)

- $\lim_{x\to -1^-} f(x)=1$
- f(-1) = 4
- $\lim_{x\to 3} f(x)=DNE$
- $\lim_{x\to -1^+} f(x)=0$
- $\lim_{x\to \infty} f(x)=5$
- $\lim_{x\to -\infty} f(x)=5$



SOLUTION

6. Given $f(x)=\begin{cases} 5-x^2, & x\le -1 \\ ax+b, & -1<x\le 4 \\ 1-\sqrt{x}, & x>4 \end{cases}$,Find the values of a and b such that the function is continuous at -1 and 4 .

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x) = f(-1)$$

$$4=-a+b$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$$

$$4a+b=-1$$

$$\begin{cases} -a+b=4 \\ 4a+b=-1 \end{cases} \Rightarrow \therefore \boxed{a=-1} \ \& \ \boxed{b=3}$$

7. Determine whether or not function $f(x)=\begin{cases} \sqrt{x+3} & , x\in(-\infty,1] \\ -x+3 & , x\in(1,7) \\ 10-3x & , x\in(7,\infty) \end{cases}$ is continuous at $x=1$ and $x=7$.

f is continuous at $x=1$, since $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x+3} = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-x+3) = 2$$

$$f(1)=2$$

f is not continuous at $x=7$, since $f(7)$ is undefined.

8. If $\lim_{x\to\infty} \frac{ax^4+1}{2x^b+x^2+1}=2$, find the value of a+b.

$$\lim_{x\to\infty} \frac{ax^4+1}{2x^b+x^2+1}$$

$$=\lim_{x\to\infty} \frac{ax^4}{2x^b}=2$$

$$\Rightarrow \boxed{b=4} \ \& \ \frac{a}{2}=2 \ or \ \boxed{a=4}$$

★ Degrees of the numerator and denominator must be equal for the $\lim_{x\to\infty} f(x)=2$

$\therefore b=4$

$$\therefore \lim_{x\to\infty} \frac{ax^4-1}{2x^4+x^2+1}=2$$

$$\lim_{x\to\infty} \frac{a-\cancel{\frac{1}{x^4}}^{\to 0}}{2+\cancel{\frac{1}{x^2}}^{\to 0}-\cancel{\frac{1}{x^4}}^{\to 0}}=2$$

$$\therefore \frac{a}{2}=2$$

$$a=4$$

9. Determine the values of p and q such that $\lim_{x\to 0} \frac{\sqrt{px+q}-3}{x}=1$.

Let $g(x)=\sqrt{px+q}-3$

★ since $g(0)=0$ we have

$$\sqrt{q}-3=0$$

$$\boxed{q=9}$$

sub q=9,

$$1=\lim_{x\to 0} \frac{\sqrt{px+9}-3}{x} \times \frac{\sqrt{px+9}+3}{\sqrt{px+9}+3}$$

$$1=\lim_{x\to 0} \frac{p\cancel{x}\cancel{+9}\cancel{-9}}{\cancel{x}\left(\sqrt{px+9}+3\right)}$$

$$1=\frac{p}{6}$$

$$\boxed{p=6}$$

★ For $\lim_{x\to 0} \frac{g(x)}{h(x)}$ to exist, a direct substitution of $x=0$ must result in $\frac{0}{0}$ so that rationalizing can be used to eliminate the indeterminate form. For this to happen, $\therefore g(0)=0$

SOLUTION

10. Jack and Cole are throwing a Frisbee in their backyard. When Jack throws to Cole, the height, in metres, of the Frisbee above the ground after t seconds is described by the function $h(t) = -4.9t^2 + 10.8t + 1$.

a) Determine the average speed of the Frisbee between $t = 1$ second and $t = 3$ seconds.

$$\text{AROC} = \frac{h(3) - h(1)}{3 - 1}$$
$$= \frac{-10.7 - 6.9}{2}$$
$$= -8.8 \text{ m/s}$$

b) Determine the instantaneous speed of the Frisbee when $t = 1$ second.

$$\text{IROC} = \lim_{h \to 0} \frac{-4.9(1+h)^2 + 10.8(1+h) + 1 - 6.9}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{-4.9} - 9.8h - 4.9h^2 \cancel{+10.8} + 10.8h \cancel{+1 - 6.9}}{h}$$
$$= \lim_{h \to 0} \frac{h - 4.9h^2}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(1 - 4.9h)}{\cancel{h}}$$
$$= 1 \text{ m/s}$$

Introduction to Derivatives

The Derivative

In our study of limits and rates of change, we saw that the slope of the tangent line to the graph of y = f(x) at the point(a,f(a)) is calculated by finding the limit of the difference quotient, and this is defined as

$$\boxed{\lim_{h\to 0}\frac{\Delta y}{\Delta x}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

provided this limit exists.

The slope of the tangent line is used to find the instantaneous rate of change of y with respect to x at x=a.

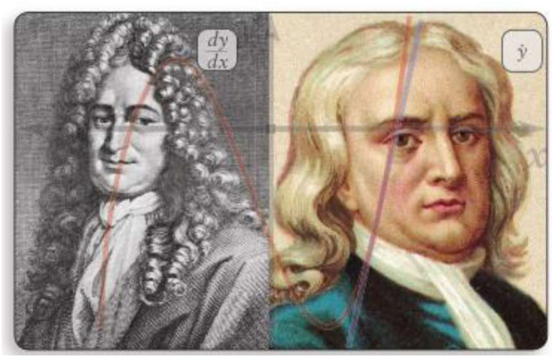
In calculus, this limit is called **the derivative** of f(x) at x=a. The process of finding the derivative is called differentiation.

By definition, the derivative, f'(x), of function f(x) for any value x in the domain of f is thus

$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$ if the limit exists

In function notation (Joseph-Louis Lagrange, 1736-1813), the derivative of the function f with respect to x is symbolized as f'(x) and pronounced "f prime of x."

In Leibniz notation (Gottfried Wilhelm Leibniz, 1684), the derivative of y is symbolized as $\frac{dy}{dx}$ and pronounced "dee y by dee x." We can also use the short form y', pronounced "y prime."



The domain of the derivative function depends on whether the value of the limit exists for all values within the domain of the original function.

This definition of derivative of f(x) is called the First Principle of Derivatives.

First Principles

$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$, if the limit exists.

- The process is called differentiating a function (or differentiation).
- The method of first principles means to use the formula above.
- The domain of $f'(x)$ depends on where the limit exists.
- It is a subset of the domain $f(x)$.
- **A derivative is a function. A collection of tangent values! (not a tangent)**

A few more notations for derivatives come from Leibniz, called **Leibniz notation**.

(i) $f'(x)$ (ii) $\frac{d}{dx}f(x)$ (iii) $\frac{dy}{dx}$ **(Leibniz's)** (iv) y'

Ex1: Differentiate the function $y = 2x^2 + 6$ using first principles.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h)^2 + 6 - (2x^2+6)}{h}$$

$$= \lim_{h\to 0} \frac{2(x^2 + 2xh + h^2) + 6 - 2x^2 - 6}{h}$$

$$= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 6 - 2x^2 - 6}{h}$$

$$= \lim_{h\to 0} \frac{h(4x+2h)}{h}$$

$$= 4x$$

Ex2: If $f(x) = \sqrt{3x}$, find f'.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)} - \sqrt{3x}}{h} \cdot \frac{\sqrt{3(x+h)} + \sqrt{3x}}{\sqrt{3(x+h)} + \sqrt{3x}}$$

$$= \lim_{h\to 0} \frac{3(x+h) - 3x}{h(\sqrt{3(x+h)} + \sqrt{3x})}$$

$$= \lim_{h\to 0} \frac{3h}{h(\sqrt{3(x+h)} + \sqrt{3x})} \quad \leftarrow \text{Do not expand !!}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(x+h)} + \sqrt{3x}}$$

$$= \frac{3}{2\sqrt{3x}}$$

Ex3: Function $f(x) = 2x^2 + ax + b$ is given. Find the constants a and b such that $f'(-1) = 2$ and the graph passes through the point $(1,6)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h)^2 + a(x+h) + b - (2x^2 + ax + b)}{h}$$

$$= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + ax + ah + b - 2x^2 - ax - b}{h}$$

$$= \lim_{h\to 0} \frac{h(4x + 2h + a)}{h}$$

$$= 4x + a$$

$\because f'(-1) = 2$ and $f'(x) = 4x + a$

$\therefore 4(-1) + a = 2$

$-4 + a = 2$

$\boxed{a = 6}$

$\because f(1) = 6$ and $f(x) = 2x^2 + 6x + b$

$\therefore 6 = 2(1)^2 + 6(1) + b$

$6 = 8 + b$

$\boxed{b = -2}$

Ex 4: Find the equation of the tangent to the curve $y = 8x - 5x^2$ at $x = 2$.

1) Find the derivative

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{[8(x+h)-5(x+h)^2]-(8x-5x^2)}{h}$$

$$= \lim_{h\to 0} \frac{8x+8h-5x^2-10xh-5h^2-8x+5x^2}{h}$$

$$= \lim_{h\to 0} \frac{8h-10xh-5h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(8-10x-5h)}{h}$$

$$= \lim_{h\to 0} 8-10x-5h$$

$$= 8-10x$$

2) Find the slope at $x=2$

$$\left.\frac{dy}{dx}\right|_{x=2} = 8-10(2)$$

$$= -12$$

3) Find the tangent point to the curve

At $x=2$, $y=8(2)-5(2)^2$
$= -4$

$\therefore$ The tangent point is $(2,-4)$.

4) Find the equation

$y-y_1 = m(x-x_1)$

$y-(-4) = (-12)[x-(2)]$

$y+4 = -12x+24$

$y = -12x+20$

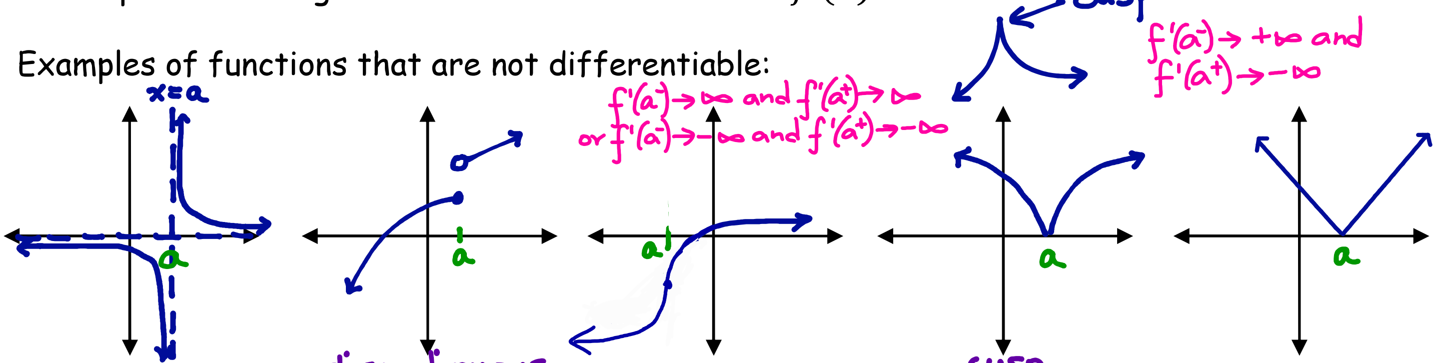
$\therefore$ The equation of the tangent at $x=2$ is $y=-12x+20$.

Differentiable Functions

The process of finding the derivative of a function is called **differentiation**. If $f'(a)$ exists, then the function f is said to be differentiable at $x = a$. $f'(a^-)=f'(a^+)$

Generally, a function is differentiable if it is continuous and if it has no vertical tangents or abrupt, discontinuous changes in slope. If a function is not differentiable, this means the slope of the tangent at $x = a$ is not defined or $f'(a) = DNE$

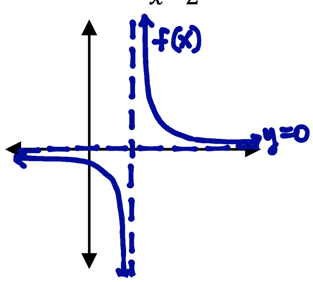
Examples of functions that are not differentiable:



vertical asymptote — discontinuous — vertical tangent — cusp $f'(a^-)\to-\infty$ and $f'(a^+)\to\infty$ — corner $f'(a^-)\to L_1$ and $f'(a^+)\to L_2$, $L_1\neq L_2$

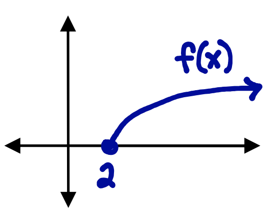
Ex. 4. Explain why the following functions are not differentiable at $x = 2$.

a) $f(x) = \frac{1}{x-2}$



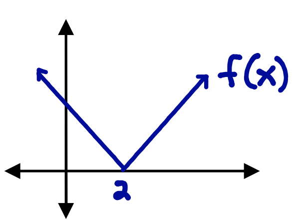
The function is not defined at x=2 since it has a vertical asymptote.

b) $f(x) = \sqrt{x-2}$



The function is not defined for $x<2$. Therefore, the limit as $x\to2$ from the left does not exist.

c) $f(x) = |x-2|$



This function has a corner at x=2 when the slope of f(x) changes abruptly from negative to positive. $\therefore f'(2)$ does not exist.

Practice:

1. A ball is tossed up in the air so that its position ***d*** in metres at time ***t*** seconds is given by $d(t) = -5t^2 + 30t + 2$.

a) What is the average velocity for the interval $[4,5]$ and $[4,4.1]$

b) What is the instantaneous velocity of the ball at $t = 4$?

a) $AROC = \dfrac{d(5) - d(4)}{5 - 4}$

$= [-5(5)^2 + 30(5) + 2] - [-5(4)^2 + 30(4) + 2]$

$= 27 - 42$

$= -15$

∴ Average velocity is -15m/s

b) $AROC = \dfrac{d(4.1) - d(4)}{4.1 - 4}$

$= \dfrac{[-5(4.1)^2 + 30(4.1) + 2] - [-5(4)^2 + 30(4) + 2]}{0.1}$

$= \dfrac{40.95 - 42}{0.1}$

$= -10.5$ ∴ Average velocity is -10.5m/s

2. An oil tank is being drained. The volume V in litres, of oil remains in the tank after time t, in minutes, is represented by the function $v(t) = 60(25 - t^2),\ t \in [0,25]$.

a) Determine the average rate of change from $t = 5$ to $t = 15$.

b) Determine the instantaneous velocity at $t = 10$ using limits.

$AROC = \dfrac{V(15) - V(5)}{15 - 5}$

$= \dfrac{60(25 - 15^2) - 60(25 - 5^2)}{15 - 5}$

$= \dfrac{-12000 - 0}{10}$

$= -1200$

∴ Average rate of change is -1200 L/min

1b) $d(t) = -5t^2 + 30t + 2$

$$d'(t) = \lim_{\Delta t \to 0} \frac{d(t+\Delta t) - d(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{-5(t+\Delta t)^2 + 30(t+\Delta t) + \cancel{2} - (-5t^2 + 30t + \cancel{2})}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{-5t^2} - 10t(\Delta t) - 5(\Delta t)^2 + \cancel{30t} + 30\Delta t + \cancel{5t^2} - \cancel{30t}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(-10t - 5\Delta t + 30)}{\cancel{\Delta t}}$$

$$= -10t + 30$$

$$d'(4) = -10(4) + 30$$

$$= -40 + 30$$

$$= -10$$

$\therefore$ Velocity of the ball is -10 m/s

2b) $v(t) = 1500 - 60t^2$

$$v'(10) = \lim_{h \to 0} \frac{1500 - 60(10+h)^2 - (1500 - 60(10)^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1500} - \cancel{6000} - 1200h - 60h^2 - \cancel{1500} + \cancel{6000}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-1200 - 60h)}{\cancel{h}}$$

$$= -1200$$

$\therefore$ Velocity is -1200 L/min

3*. Determine the coordinates of the point(s) on the graph of $y = 3x - \frac{1}{x}$ at which the slope of the tangent is 7.

$$y = 3x - \frac{1}{x}$$

$$m_{tangent} = y' = \lim_{h \to 0} \frac{3(x+h) - \frac{1}{x+h} - 3x + \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x} + 3h - \frac{1}{x+h} - \cancel{3x} + \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{3h(x+h)(x) - \cancel{x} + \cancel{x} + h}{h(x)(x+h)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}[3x(x+h) + 1]}{\cancel{h}(x)(x+h)}$$

$$= \frac{3x^2 + 1}{x^2}$$

$$\therefore 7 = \frac{3x^2+1}{x^2}$$

$$7x^2 - 3x^2 = 1$$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

At $x = \frac{1}{2}$, $y = 3\left(\frac{1}{2}\right) - \frac{1}{\frac{1}{2}}$

$$= \frac{3}{2} - 2$$

$$= -\frac{1}{2}$$

At $x = -\frac{1}{2}$, $y = 3\left(-\frac{1}{2}\right) - \frac{1}{-\frac{1}{2}}$

$$= -\frac{3}{2} + 2$$

$$= \frac{1}{2}$$

$\therefore$ The points are $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Example #1 For the function $f(x)=3x^2+2$, determine the slope of a tangent at x = 1

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{3(x+h)^2+2-(3x^2+2)}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{3x^2}+6xh+3h^2\cancel{+2}\cancel{-3x^2}\cancel{-2}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(6x+3h)}{\cancel{h}}$$

$$=6x$$

$$\therefore f'(1)=6(1)$$

$$=6$$

$\therefore$ The slope of the tangent is 6.

Example #2 Find the equation of the tangent to the curve $y=\frac{1}{x-1}$ at (2,1).

$$\left.\frac{dy}{dx}\right|_{x=2}=\lim_{h\to 0}\frac{\frac{1}{2+h-1}-\frac{1}{2-1}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{1+h}-1}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{1}\cancel{-1}-\cancel{h}^{1}}{\cancel{h}_{1}(1+h)}$$

$$=-1$$

At (2,1), $y-y_1=m(x-x_1)$

$$y-1=-1(x-2)$$

$$y=-x+2+1$$

$$y=-x+3$$

Example #3 Find the **equation of tangent line** to the curve $f(x)=\sqrt{3-x}$ at x = -1 on the curve.

$$f'(-1)=\lim_{h\to 0}\frac{\sqrt{3-(-1+h)}-\sqrt{3-(-1)}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{4-h}-2}{h}\cdot\frac{\sqrt{4-h}+2}{\sqrt{4-h}+2}$$

$$=\lim_{h\to 0}\frac{\cancel{4}-\cancel{h}^{-1}-\cancel{4}}{\cancel{h}_{1}(\sqrt{4-h}+2)}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{4-h}+2}$$

$$=\frac{-1}{4}$$

$$f(-1)=\sqrt{3-(-1)}$$
$$=2$$

$$\therefore y-2=-\frac{1}{4}(x+1)$$
$$y=-\frac{1}{4}x-\frac{1}{4}+2$$
$$y=-\frac{1}{4}x+\frac{7}{4}$$

Example #4 Determine the coordinates of the point where the tangent to the function $y=x^3$ is perpendicular to the line $-3x-4y+12=0$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}}$$

$$=3x^2$$

$$-3x-4y+12=0$$
$$4y=-3x+12$$
$$y=-\frac{3}{4}x+3$$
$$m=-\frac{3}{4}$$
$$m_\perp=\frac{4}{3}$$

$$\therefore 3x^2=\frac{4}{3}$$
$$x^2=\frac{4}{9}$$
$$x=\pm\frac{2}{3}$$

At $x=\frac{2}{3}$, $y=\left(\frac{2}{3}\right)^3$
$$=\frac{8}{27}$$

At $x=-\frac{2}{3}$, $y=\left(-\frac{2}{3}\right)^3$
$$=\frac{-8}{27}$$

$\therefore$ The points are $\left(\frac{2}{3},\frac{8}{27}\right)$ and $\left(\frac{-2}{3},\frac{-8}{27}\right)$

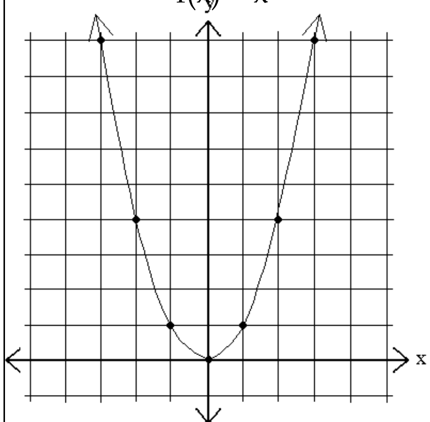
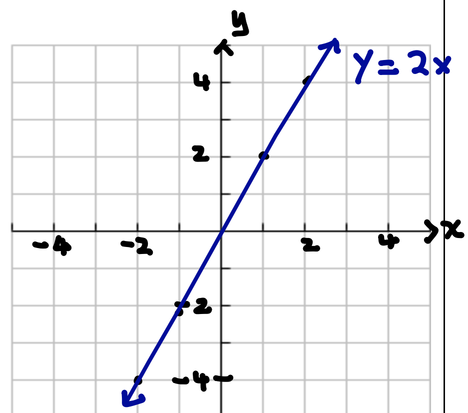
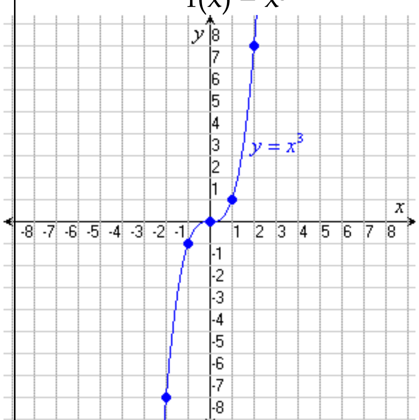
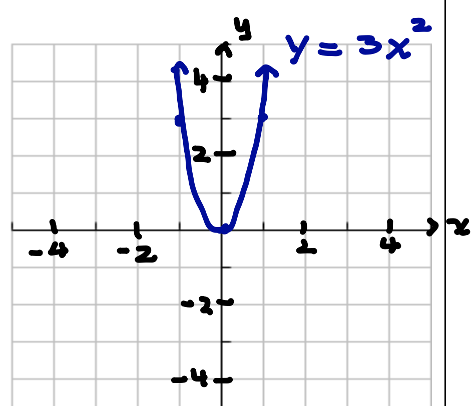
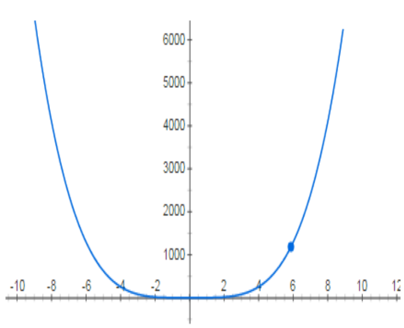
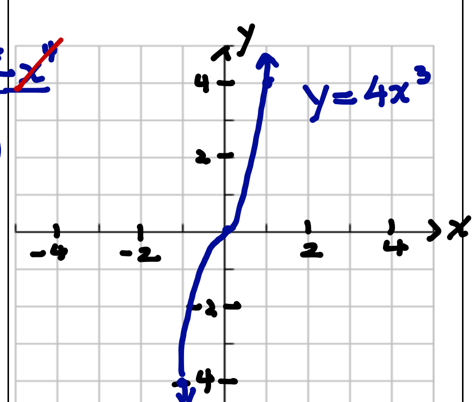
Derivatives of Polynomial Functions
By First Principles

Investigating Differentiation Shortcuts

Investigation #1: Powers

Conjecture about the Derivative of Power Functions

1) Complete the following chart to summarize your findings above.

Function f(x)	Equation of Derivative f′(x) (using the first principles)	Shape/Type of Graph of Derivative y = f′(x)
f(x) = x² 	$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$ $=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$ $=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}$ $=2x$	
f(x) = x³ 	$f'(x)=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$ $=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2-\cancel{x^3}}{h}$ $=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh)}{\cancel{h}}$ $=3x^2$	
f(x) = x⁴ 	$f'(x)=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$ $=\lim_{h\to 0}\frac{\cancel{x^4}+4x^3h+6x^2h^2+4xh^3+h^4-\cancel{x^4}}{h}$ $=\lim_{h\to 0}\frac{\cancel{h}(4x^3+6x^2h+4xh^2+h^3)}{\cancel{h}}$ $=4x^3$	

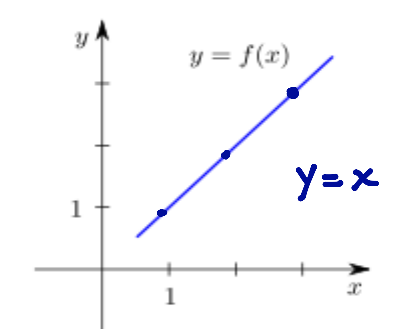
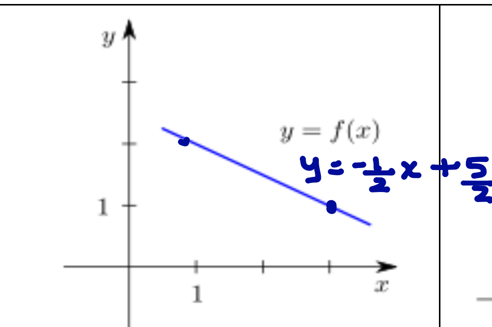
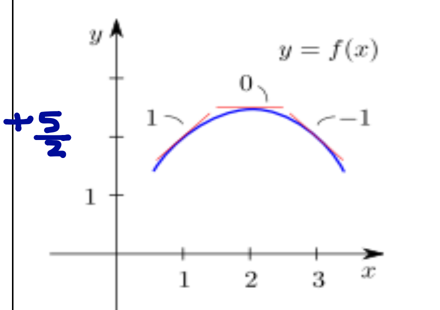
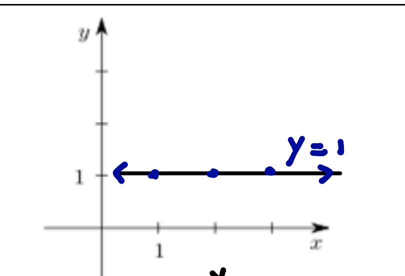
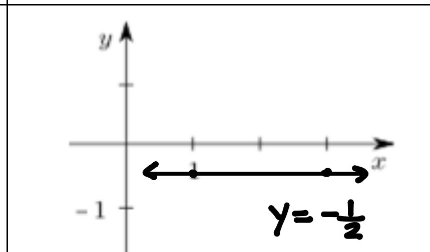
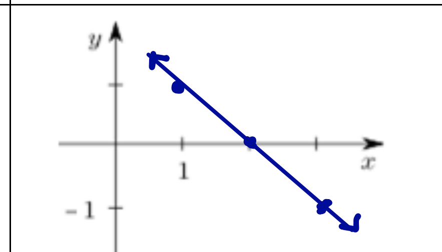
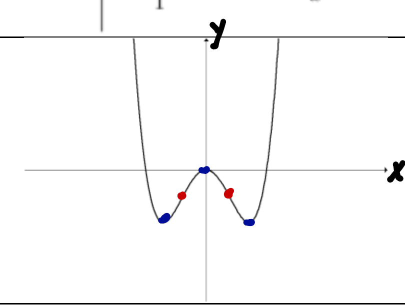
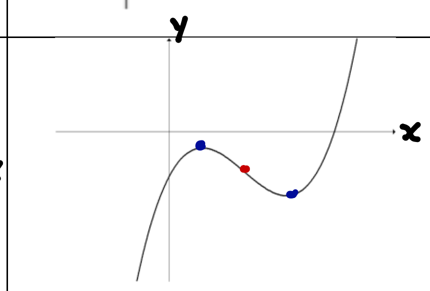
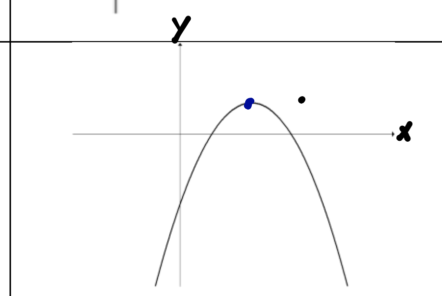
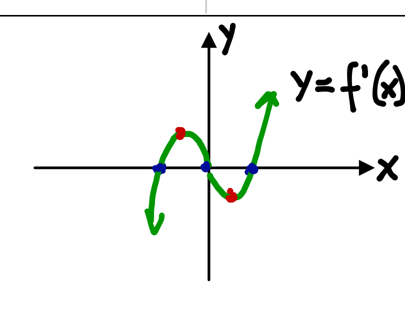
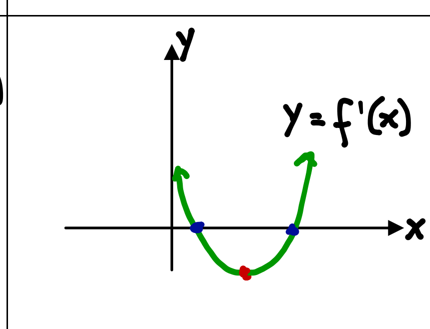
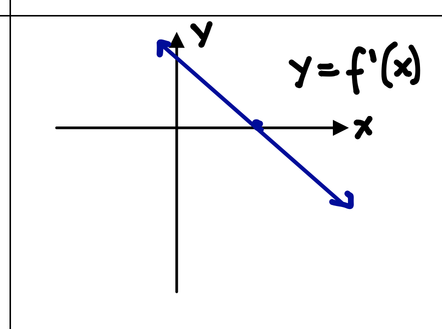
Derivatives of Polynomial Functions
By First Principles

Relating graph of function to graph of derivative

We give a series of examples with the graph of a function on the top and the graph of its derivative on the bottom.

To begin, we recall two basic facts about the derivative $f'(x)$ of a function $f(x)$:

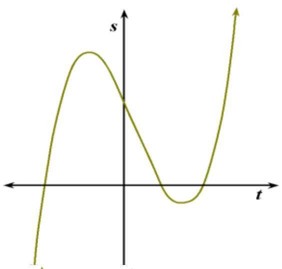
1. The value $f'(a)$ is the slope of the tangent to the graph of the function $f(x)$ at the point where $x=a$.
2. $f'(x)$ is a function of x: the slope at a point on the graph depends on the –coordinate of that point.

Graph of f(x)			
Graph of f'(x)			
Graph of f(x)			
Graph of f'(x)			

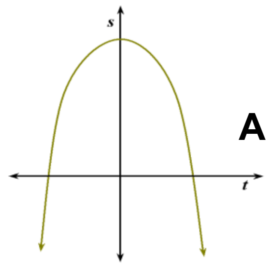
- If the slope of the tangent is negative, then the derivative graph is below the x-axis.
- If the slope of the tangent is positive, then the derivative graph is above the x-axis.
- If the slope of the tangent is zero, then the derivative graph is on the x-axis.
- The point of inflection of the polynomial function will be the max. or min point of the derivative graph
- The slope of the tangent is zero at the max. or min. point

Derivatives of Polynomial Functions
By First Principles

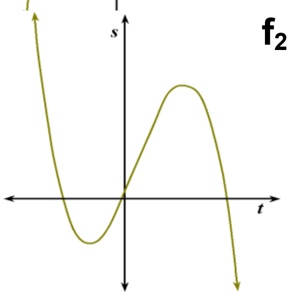
You Try! Match each function on the left with its derivative graph on the right.



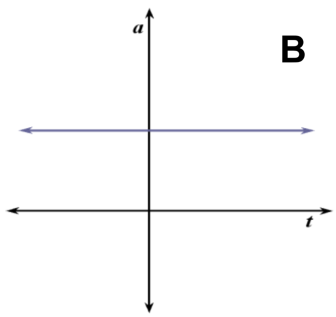
f1



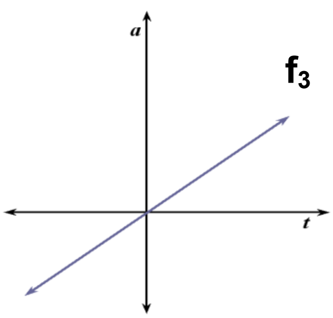
A



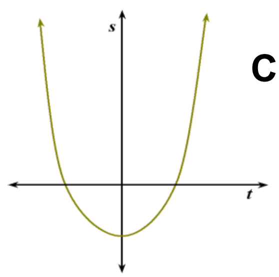
f2



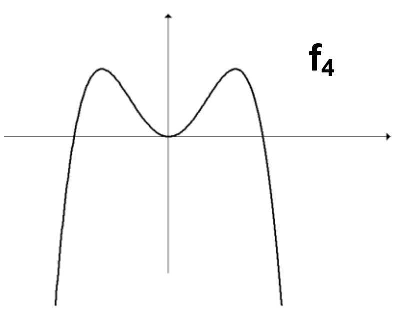
B



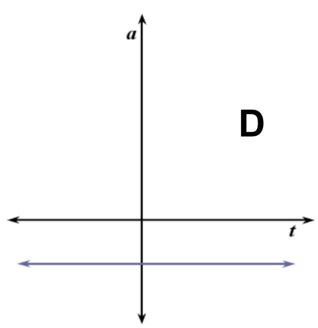
f3



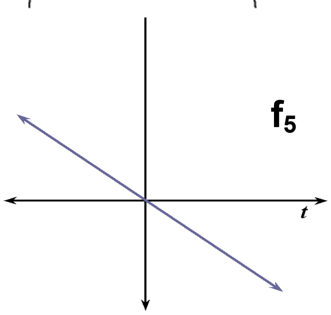
C



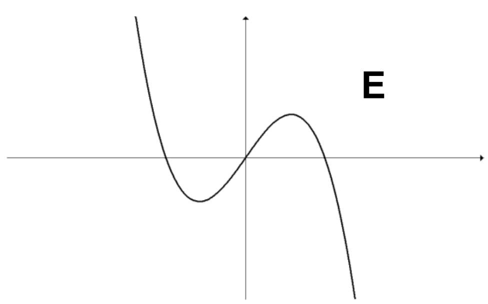
f4



D



f5



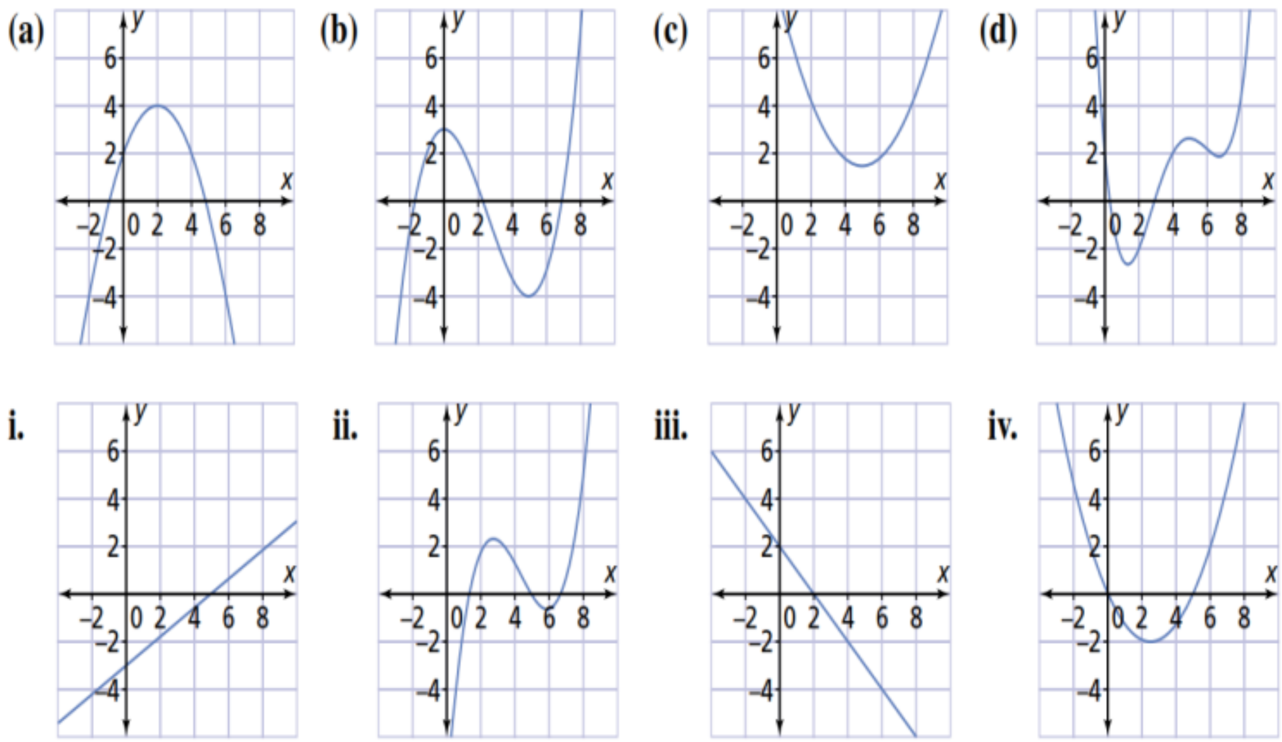
E

Function	Derivative graph
f_1	C
f_2	A
f_3	B
f_4	E
f_5	D

Derivatives of Polynomial Functions
By First Principles

Practice:

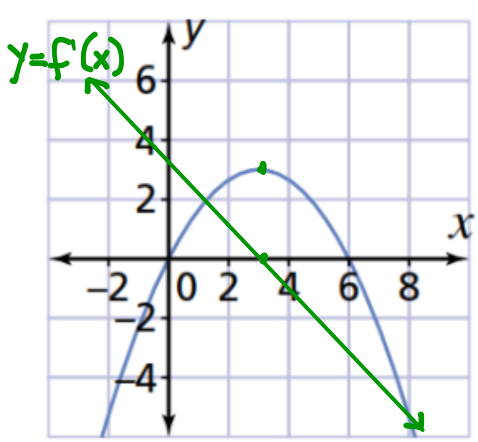
1. The graphs of four functions are drawn in the top row. The graphs of their derivatives are drawn in the bottom row. Match each function with its derivative.



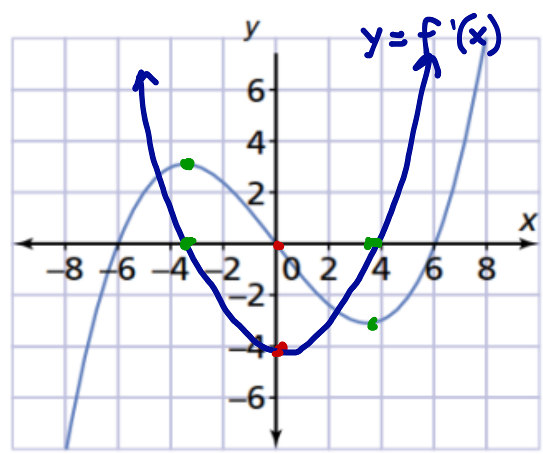
C D A B

2. For the graph of each function, estimate and graph the derivative function.

(a)



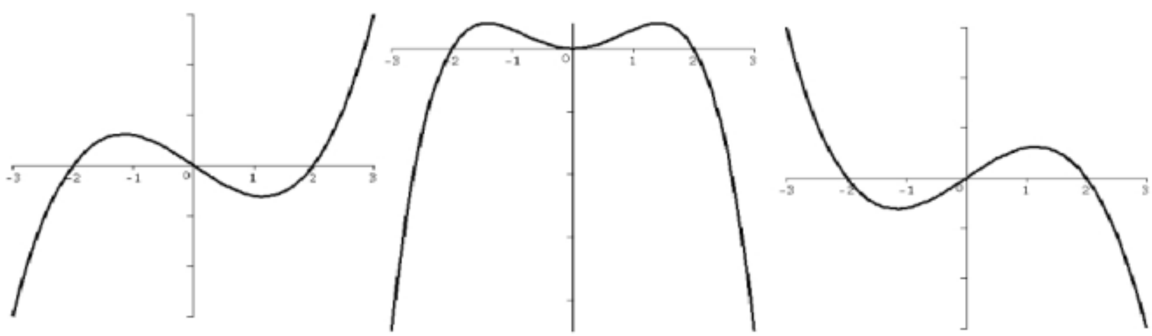
(b)



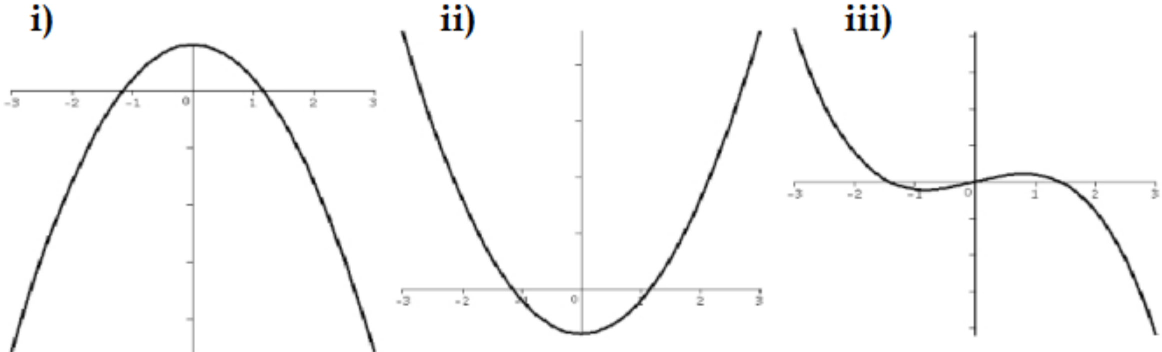
Derivatives of Polynomial Functions
By First Principles

3. The graphs of three functions are drawn in the top row. The graphs of their derivatives are drawn in the bottom row. Match each function with its derivative.

a) b) c)



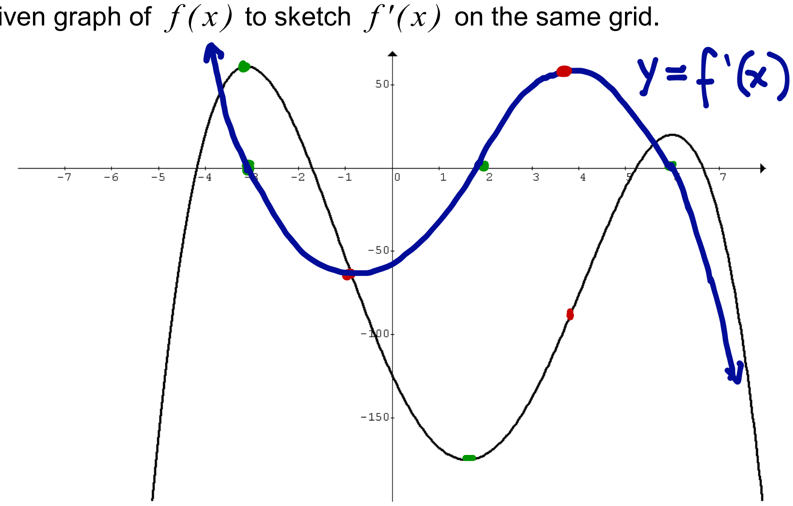
i) ii) iii)



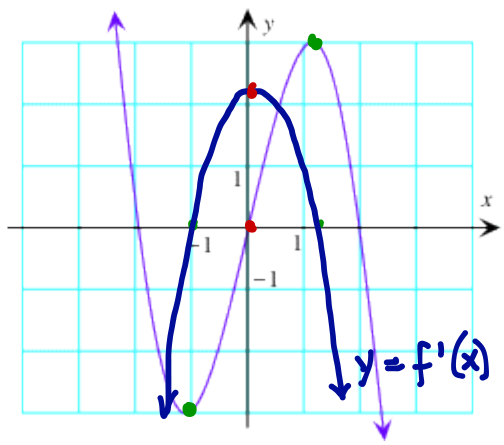
C A B

4. Use the given graph of $f(x)$ to sketch $f'(x)$ on the same grid.

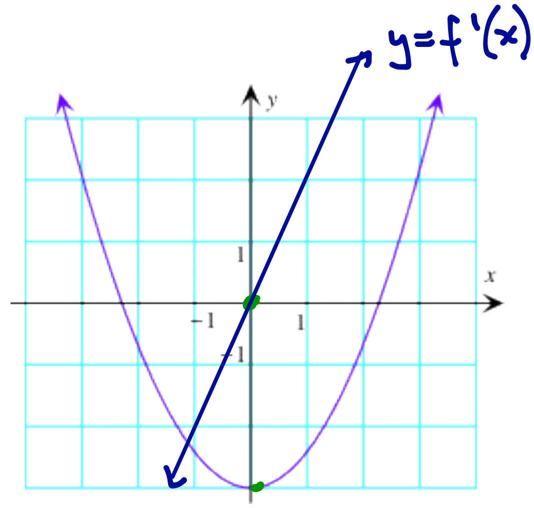
a)



b)



c)



Practice

1. A curve is defined by $f(x) = \boldsymbol{a}\sqrt[3]{x} + \boldsymbol{b}$, where **a** and ***b*** are constants. Given that $m_t = 5$ at the point (1, 9) , determine the values of *a* and *c*.

2. Determine the equation of the line that is tangent to the graph $f(x) = \sqrt{x+1}$ and is perpendicular to the line $6x + y - 5 = 0$

3. Let P be any point in the first quadrant on the hyperbola $y = \dfrac{1}{x}$. The tangent to the hyperbola meets the *x*-axis at A and the *y*-axis at B. If O is the origin, determine the area of ΔAOB

4. Determine the points on the curve $f(x) = 1 - \dfrac{1}{x}$ where the tangent line is perpendicular to the line $4x + y - 1 = 0$.

1. A curve is defined by $f(x) = \mathbf{a}\sqrt[3]{x} + \mathbf{b}$, where **a** and ***b*** are constants. Given that $m_t = 5$ at the point (1, 9), determine the values of *a* and *c*.

$$f(1) = 9 \rightarrow a + b = 9 \rightarrow b = 9 - a \quad (1)$$

Slope of the tangent at $x=1$:

$$5 = \lim_{h \to 0} \frac{a\sqrt[3]{1+h} + b - (a+b)}{h}$$

$$5 = a\lim_{h \to 0} \frac{\sqrt[3]{1+h} - 1}{h}$$

Let $\sqrt[3]{1+h} = u$

$h = u^3 - 1$

As $h \to 0, u \to 1$

$$5 = a\lim_{u \to 1} \frac{u-1}{u^3-1}$$

$$5 = a\lim_{u \to 1} \frac{\cancel{u-1}}{\cancel{(u-1)}(u^2+u+1)}$$

$$5 = \frac{a}{3} \Rightarrow \boxed{a = 15} \xrightarrow{\text{sub. into (1)}} \boxed{b = -6}$$

2. Determine the equation of the line that is tangent to the graph $f(x) = \sqrt{x+1}$ and is perpendicular to the line $6x + y - 5 = 0$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h+1} - \sqrt{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{x+h+1-x-1}{h\left[\sqrt{x+h+1} + \sqrt{x+1}\right]}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left[\sqrt{x+h+1} + \sqrt{x+1}\right]}$$

$$= \frac{1}{2\sqrt{x+1}}$$

$6x + y - 5 = 0$

$m_t = -6$

$m_\perp = \frac{1}{6}$

$f'(x) = m_\perp$:

$$\frac{1}{2\sqrt{x+1}} = \frac{1}{6}$$

$\sqrt{x+1} = 3$

$x + 1 = 9$

$x = 8$

$f(8) = \sqrt{(8)+1}$

$= 3$ $\therefore$ Tangent point at $(8,3)$.

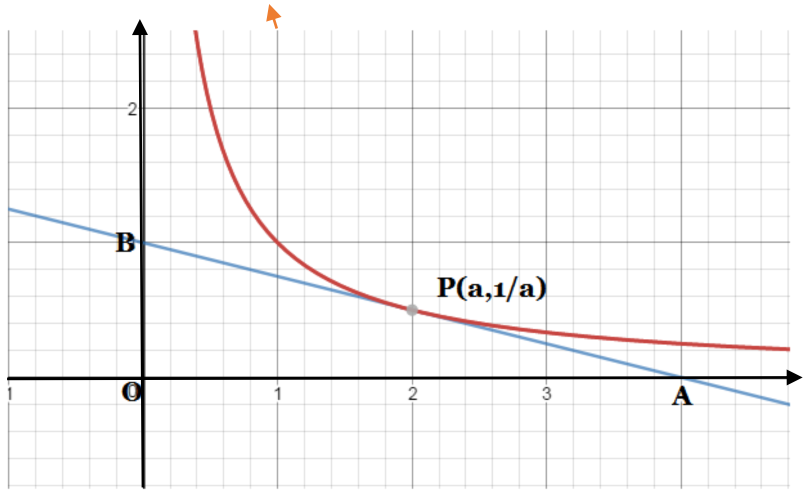
Equation of the line perpendicular to the line $y = -6x + 5$ at point (8,3) is

$y - 3 = \frac{1}{6}(x - 8)$ **or**

$y = \frac{1}{6}x + \frac{5}{4}$

3. Let P be any point in the first quadrant on the hyperbola $y = \frac{1}{x}$. The tangent to the hyperbola meets the x-axis at A and the y-axis at B. If O is the origin, determine the area of ΔAOB.

Let $\left(a, \frac{1}{a}\right)$ represent the tangency point



$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{a - a - h}{ah(a+h)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}a(a+h)}$$

$$= \frac{-1}{a^2}$$

Equation of the tangent line at point $\left(a, \frac{1}{a}\right)$ is

$$y - \frac{1}{a} = \frac{-1}{a^2}(x - a)$$

x-int: $y = 2a$, $A(2a, 0)$

y-int: $y = \frac{2}{a}$, $B(0, \frac{2}{a})$

Area of $\Delta OAB = \dfrac{(2a)(\frac{2}{a})}{2}$

$$\boxed{A_{\Delta OAB} = 2 \text{ units}^2}$$

4. Determine the points on the curve $f(x)=1-\frac{1}{x}$ where the tangent line is perpendicular to the line $4x+y-1=0$.

$$f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left[1-\frac{1}{x+h}\right]-\left[1-\frac{1}{x}\right]}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{x}-\frac{1}{x+h}}{h}$$

$$=\lim_{h\to 0}\frac{x+h-x}{hx(x+h)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}x(x+h)}$$

$$f'(x)=\frac{1}{x^2}$$

$$y=1-4x \rightarrow m=-4 \therefore m_{\perp}=\frac{1}{4}$$

$f'(x)=m_{\perp}$:

$$\frac{1}{x^2}=\frac{1}{4}$$

$$x^2=4$$

$$x=\pm 2$$

$$f(-2)=1-\frac{1}{(-2)} = \frac{3}{2}$$

$$f(2)=1-\frac{1}{(2)} = \frac{1}{2}$$

$\therefore \left(2,\frac{1}{2}\right), \left(-2,\frac{3}{2}\right)$ are the required points.

REVIEW

1. Find the values of a and b if the function $f(x)=\begin{cases} -x^2 & x<0 \\ ax+b & 0\le x<1 \\ \sqrt{x+3} & x\ge 1 \end{cases}$ is continuous.

2. Let $f(x)=\dfrac{x^2-9}{x}$. State the discontinuities of $f(x)$ and find their type.

3. A designer is experimenting with a cylindrical can with a fixed height of 15 cm. find the rate of change of volume with respect to radius when the radius is 4 cm. the volume of a cylinder is $V=\pi r^2 h$.

4. Let $f(x)=ax^2+bx+c$. Find a,b and c so that the tangent to the graph $y=f(x)$ has slope 16 where $x=2$ and has x-intercepts (0,0) and (8,0)

5. Is the following statement true? Justify your answer.

It is possible for a limit to exist even though the function may not be continuous at the point of question.

6. Function $f(x)=\begin{cases} \dfrac{x^2-1}{x+1} & \text{if} \quad x\ne -1 \\ -2k+1 & \text{if} \quad x=-1 \end{cases}$ is continuous at $x=-1$.Find the value of k.

7. At what point on the parabola $y=3x^2$ is the slope of the tangent equal to 24?

8. Find the points on the curve $y=1-\dfrac{1}{x}$ where the tangent line is perpendicular to the line $y=1-4x$.

9. Find the average rate of change of the function $f(x)=\dfrac{5\sqrt{x}}{x+2}$ between $x=1$ and $x=4$ .What is the instantaneous rate of change at x= 1.

10. Evaluate the following limits, if they exist. Show your work.

a) $\lim\limits_{x\to\frac{2}{3}y}\dfrac{27x^3-8y^3}{9x^2-4y^2}$ b) $\lim\limits_{x\to\infty}7^{-(x-3)}$ c) $\lim\limits_{x\to 0}5^{x-3}$ d) $\lim\limits_{x\to 2}\dfrac{x^3-x^2-4x+4}{x^2-4}$

e) $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x+x^2}-2}{x}$ f) $\lim\limits_{x\to\infty}\dfrac{\left(\frac{2}{3}\right)^x-2}{\left(\frac{2}{3}\right)^x+2}$ g) $\lim\limits_{x\to 4^+}\dfrac{x-4}{\sqrt{x}-2}$ h) $\lim\limits_{x\to 2}\dfrac{\sqrt{x^2+5}-\sqrt{x+7}}{x-2}$

i) $\lim\limits_{x\to 2}\dfrac{\sqrt[3]{2x-5}+1}{x-2}$ j) $\lim\limits_{x\to\infty}\dfrac{(2x+3)^2}{5-2x-5x^2}$ k) $\lim\limits_{x\to 2}\dfrac{\frac{1}{x-5}+\frac{1}{3}}{x-2}$

11. The position of a particle moving along the x-axis is given by $s(t) = t^2 - 5t + 4$ where t is the elapsed time in seconds.

(a) What is the position of the particle after 2 seconds?

(b) Calculate the average velocity of the particle from t = 2s to t = 5s.

(c) Calculate the instantaneous velocity of the particle when t = 2s.

12. Larry is driving over the speed limit on wet, slippery roads. The road can be modelled by the curve $f(x) = x^2$. As he travels from left to right, his car begins to slide out of the curve in a straight line at the point (-1, 1). A stupid cat is sitting at the point (3, -7). Will Larry run over the cat?

13. Find the point(s) in exact value where the functions $y = 2x^3 + 10x - 1$ and $y = -\frac{4}{x}$ contain the same slope.

14. Determine the values of b and c in the function $f(x) = x^2 - 3bx + (c+2)$, if $f(x)$ has an x-intercept at $x = 1$ and a $f'(3) = 0$.

Unit 2-Review Solutions

1. Find the values of a and b if the function $f(x)=\begin{cases}-x^2 & x<0\\ ax+b & 0\le x<1\\ \sqrt{x+3} & x\ge 1\end{cases}$ is continuous.

$$\lim_{x\to 0^-}\left(-x^2\right)=\lim_{x\to 0^+}(ax+b)=f(0)$$
$$0=b$$
$$\lim_{x\to 1^-}(ax+b)=\lim_{x\to 1^+}\left(\sqrt{x+3}\right)=f(1)$$
$$a+b=2\to \text{since } b=0\ , a=2$$

2. Let $f(x)=\dfrac{x^2-9}{x}$. State the discontinuities of $f(x)$ and find their type.

At x=0, infinite discontinuity.

3. A designer is experimenting with a cylindrical can with a fixed height of 15 cm. find the rate of change of volume with respect to radius when the radius is 4 cm. The volume of a cylinder is $V=\pi r^2 h$.

$$V'(r)=\lim_{h\to 0}\frac{V(r+h)-V(r)}{h}$$
$$=\lim_{h\to 0}\frac{\pi(r+h)^2 H-\pi r^2 H}{h}$$
$$=\lim_{h\to 0}\frac{\pi H\left[(r+h)^2-r^2\right]}{h}$$
$$=\pi H\lim_{h\to 0}\frac{\left[\cancel{r^2}+2rh+h^2-\cancel{r^2}\right]}{h}$$
$$=\pi H\lim_{h\to 0}\frac{\cancel{h}(2r+h)}{\cancel{h}}$$
$$=2\pi r H$$
$$at\ \ H=15cm\ \ and\ \ r=4cm: V'(4)=2\pi(4cm)(15cm)=120\pi cm^2$$

4. Let $f(x)=ax^2+bx+c$. Find a,b and c so that the tangent to the graph $y=f(x)$ has slope 16 where $x=2$ and has x-intercepts (0,0) and (8,0).

$$f'(x)=2ax+b$$
$$f'(2)=16\to 2a(2)+b=16\to 4a+b=16 \quad (1)$$
$$f(0)=0\to c=0$$
$$f(8)=0\to 64a+8b=0\to 8a+b=0 \quad (2)$$
$$\begin{cases}4a+b=16\\ 8a+b=0\end{cases}\Rightarrow a=-4\ \&\ b=32$$

$$f'(x)=\lim_{h\to 0}\frac{a(x+h)^2+b(x+h)+c-ax^2-bx-c}{h}$$
$$=\lim_{h\to 0}\frac{ax^2+2axh+ah^2+bx+bh+c-ax^2-bx-c}{h}$$
$$=\lim_{h\to 0}\frac{2axh+bh}{h}$$
$$=\lim_{h\to 0}\frac{h(2ax+b)}{h}$$
$$=2ax+b$$

5. Is the following statement true? Justify your answer.

It is possible for a limit to exist even though the function may not be continuous at the point of question.

Yes it's true. For example: $f(x)=\begin{cases} x & if \quad x\neq 0 \\ 1 & if \quad x=0 \end{cases}$ ***is not continuous at x=0, however***

$\lim_{x\to 0} f(x)=\lim_{x\to 0}(x)=0$.

6. Function $f(x)=\begin{cases} \dfrac{x^2-1}{x+1} & if \quad x\neq -1 \\ -2k+1 & if \quad x=-1 \end{cases}$ is continuous at $x=-1$ .Find the value of k.

$\lim_{x\to -1}\left(\dfrac{x^2-1}{x+1}\right)=-2k+1$ ← If $f(x)$ is continuous at $x=-1$, $\lim_{x\to -1} f(x)=f(-1)$

$\lim_{x\to -1}\dfrac{(x-1)\cancel{(x+1)}}{\cancel{x+1}}=-2k+1$

$\lim_{x\to -1}(x-1)=-2k+1$

$-2=-2k+1\rightarrow k=\dfrac{3}{2}$

7. At what point on the parabola $y=3x^2$ is the slope of the tangent equal to 24?

$y=3x^2\rightarrow y'=6x$ ← $y'=\lim_{h\to 0}\dfrac{3(x+h)^2-3x^2}{h}$

$6x=24\rightarrow x=4$ $=\lim_{h\to 0}\dfrac{3x^2+6xh+3h^2-3x^2}{h}$ $=\lim_{h\to 0}\dfrac{3h(2x+h)}{h}$

$\therefore(4,48)$ $=\lim_{h\to 0}3(2x+h)$

$=6x$

8. Find the points on the curve $y=1-\dfrac{1}{x}$ where the tangent line is perpendicular to the line $y=1-4x$.

Use First Principles ★ $y=1-\dfrac{1}{x}\rightarrow y'=\dfrac{1}{x^2}$

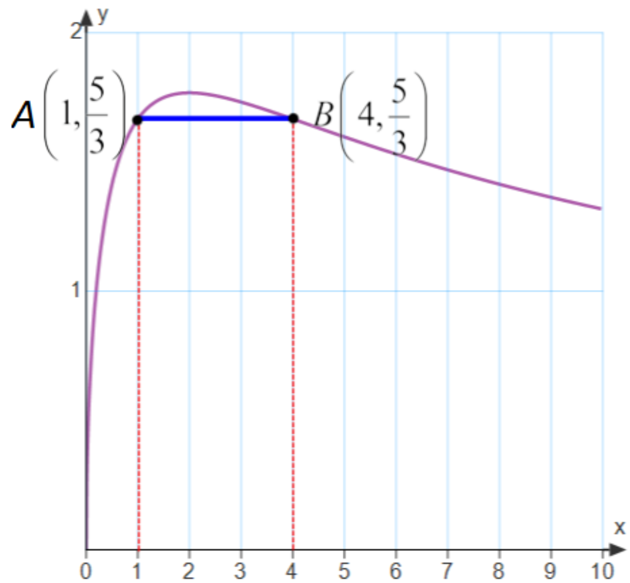
$y'=m_\perp$:

$\dfrac{1}{x^2}=\dfrac{1}{4}\rightarrow x^2=4\rightarrow x=\pm 2$

$y=1-4x\rightarrow m=-4\rightarrow m_\perp=\dfrac{1}{4}$

$\therefore\left(2,\dfrac{1}{2}\right),\left(-2,\dfrac{3}{2}\right)$ are the required points

9. Find the average rate of change of the function $f(x)=\dfrac{5\sqrt{x}}{x+2}$ between $x=1$ and $x=4$ .What is the instantaneous rate of change at x= 1.



$$\underset{[1,4]}{ARoC}=\frac{f(4)-f(1)}{4-1}$$

$$=\frac{\frac{10}{6}-\frac{5}{3}}{3}$$

$$=0$$

$$IRoC=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{5\sqrt{1+h}}{1+h+2}-\frac{5}{3}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{5\sqrt{1+h}}{h+3}-\frac{5}{3}}{h}$$

$$=5\lim_{h\to 0}\frac{3\sqrt{1+h}-(h+3)}{3h(h+3)}\times\frac{3\sqrt{1+h}+(h+3)}{3\sqrt{1+h}+(h+3)}$$

$$=5\lim_{h\to 0}\frac{9(1+h)-(h+3)^2}{3h(h+3)\left[3\sqrt{1+h}+(h+3)\right]}$$

$$=5\lim_{h\to 0}\frac{\cancel{9}+9h-h^2-6h\cancel{-9}}{3h(h+3)\left[3\sqrt{1+h}+(h+3)\right]}$$

$$=5\lim_{h\to 0}\frac{3h-h^2}{3h(h+3)\left[3\sqrt{1+h}+(h+3)\right]}$$

$$=5\lim_{h\to 0}\frac{\cancel{h}(3-h)}{3\cancel{h}(h+3)\left[3\sqrt{1+h}+(h+3)\right]}$$

$$=5\left[\frac{[3-(0)]}{3\left[(0)+3\right]\left[3\sqrt{1+(0)}+[(0)+3]\right]}\right]=\frac{5}{18}$$

10. Evaluate the following limits, if they exist. Show your work.

a) $$\lim_{x\to\frac{2}{3}y}\frac{27x^3-8y^3}{9x^2-4y^2}=\lim_{x\to\frac{2}{3}y}\frac{\cancel{(3x-2y)}\left(9x^2+6xy+4y^2\right)}{\cancel{(3x-2y)}(3x+2y)}=3y$$

b) $$\lim_{x\to\infty}7^{-(x-3)}=7^{-\infty}=0$$

c) $\lim_{x\to 0} 5^{x-3} = 5^{-3} = \frac{1}{125}$

d) $\lim_{x\to 2} \frac{x^3-x^2-4x+4}{x^2-4} = \lim_{x\to 2} \frac{\cancel{(x-2)}\left(x^2+x-2\right)}{\cancel{(x-2)}(x+2)} = 1$

e) $\lim_{x\to 0} \frac{\sqrt{4+x+x^2}-2}{x} \times \frac{\sqrt{4+x+x^2}+2}{\sqrt{4+x+x^2}+2}$

$= \lim_{x\to 0} \frac{\cancel{4}+x+x^2\cancel{-4}}{x\left(\sqrt{4+x+x^2}+2\right)}$

$= \lim_{x\to 0} \frac{\cancel{x}(1+x)}{\cancel{x}\left(\sqrt{4+x+x^2}+2\right)}$

$= \frac{1}{4}$

f) $\lim_{x\to\infty} \frac{\cancel{\left(\frac{2}{3}\right)^{x}}^{0} - 2}{\cancel{\left(\frac{2}{3}\right)^{x}}_{0} + 2} = \frac{-2}{2} = -1$

g) $\lim_{x\to 4^+} \frac{x-4}{\sqrt{x}-2} \times \frac{\sqrt{x}+2}{\sqrt{x}+2}$

$= \lim_{x\to 4^+} \frac{\cancel{(x-4)}\left(\sqrt{x}+2\right)}{\cancel{x-4}}$

$= 4$

h) $\lim_{x\to 2} \frac{\sqrt{x^2+5}-\sqrt{x+7}}{x-2} \times \frac{\sqrt{x^2+5}+\sqrt{x+7}}{\sqrt{x^2+5}+\sqrt{x+7}}$

$= \lim_{x\to 2} \frac{\left(x^2+5\right)-(x+7)}{(x-2)\left[\sqrt{x^2+5}+\sqrt{x+7}\right]}$

$= \lim_{x\to 2} \frac{x^2-x-2}{(x-2)\left[\sqrt{x^2+5}+\sqrt{x+7}\right]}$

$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}\left[\sqrt{x^2+5}+\sqrt{x+7}\right]}$

$= \frac{1}{2}$

i) $\lim_{x\to 2}\frac{\sqrt[3]{2x-5}+1}{x-2}$

Let $\sqrt[3]{2x-5}=u$ As $x\to 2,\ u\to -1$

$2x-5=u^3$

$x=\frac{u^3+5}{2}$

$$\lim_{x\to 2}\frac{\sqrt[3]{2x-5}+1}{x-2}=\lim_{u\to -1}\frac{u+1}{\frac{u^3+5}{2}-2}$$

$$=\lim_{u\to -1}\frac{2(u+1)}{u^3+5-4}$$

$$=\lim_{u\to -1}\frac{2(u+1)}{u^3+1}$$

$$=\lim_{u\to -1}\frac{2\cancel{(u+1)}}{\cancel{(u+1)}(u^2-u+1)}$$

$$=\frac{2}{3}$$

j) $\lim_{x\to\infty}\frac{(2x+3)^2}{5-2x-5x^2}$

$$=\lim_{x\to\infty}\frac{4x^2+12x+9}{-5x^2-2x+5}$$

$$=\lim_{x\to\infty}\frac{\cancel{x^2}\left(4+\cancel{\frac{12}{x}}^{0}+\cancel{\frac{9}{x^2}}^{0}\right)}{\cancel{x^2}\left(-5-\cancel{\frac{2}{x}}_{0}+\cancel{\frac{5}{x^2}}_{0}\right)}$$

$$=\frac{-4}{5}$$

or $\lim_{x\to\infty}\frac{x^2\left(2+\frac{3}{x}\right)^2}{x^2\left(\frac{5}{x^2}-\frac{2}{x}-5\right)}$

$=\frac{2^2}{-5}$

$=-\frac{4}{5}$

k) $\lim_{x\to 2}\frac{\frac{1}{x-5}+\frac{1}{3}}{x-2}$

$$=\lim_{x\to 2}\frac{3+x-5}{3(x-2)(x-5)}$$

$$=\lim_{x\to 2}\frac{\cancel{x-2}}{3\cancel{(x-2)}(x-5)}$$

$$=\frac{-1}{9}$$

11. The position of a particle moving along the x-axis is given by $s(t) = t^2 - 5t + 4$ where t is the elapsed time in seconds.

(a) What is the position of the particle after 2 seconds?

s(2) = $2^2 - 5(2) + 4 = -2$

The particle's position after 2 seconds is 2m to the left of the origin.

(b) Calculate the average velocity of the particle from t = 2s to t = 5s.

$$\begin{aligned} ARoC &= \frac{s(5) - s(2)}{5-2} \\ &= \frac{4-(-2)}{3} \\ &= 2 \text{ m/s} \end{aligned}$$

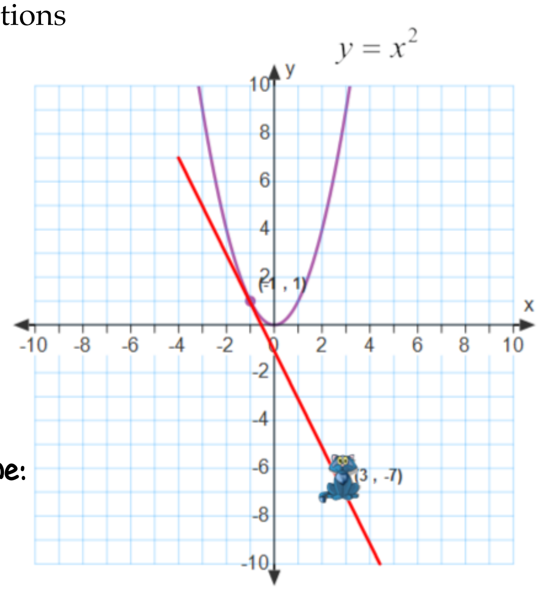
The average velocity is 2m/s.

(c) Calculate the instantaneous velocity of the particle when t = 2s.

$$\begin{aligned} s'(2) &= \lim_{h\to 0} \frac{\left[(2+h)^2 - 5(2+h) + 4\right] - (-2)}{h} \\ &= \lim_{h\to 0} \frac{4 + h^2 + 4h - 10 - 5h + 4 + 2}{h} \\ &= \lim_{h\to 0} \frac{h^2 - h}{h} \\ &= \lim_{h\to 0} \frac{\cancel{h}(h-1)}{\cancel{h}} \\ &= -1 \text{ m/s} \end{aligned}$$

The instantaneous velocity is -1m/s

12. Larry is driving over the speed limit on wet, slippery roads. The road can be modelled by the curve f(x) = x^2. As he travels from left to right, his car begins to slide out of the curve in a straight line at the point (-1, 1). A stupid cat is sitting at the point (3, -7). Will Larry run over the cat?



$f(x) = x^2$

Slope of tangent:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \ a = -1$$

$$m = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$y = -2x - 1$

$$m = \lim_{h \to 0} \frac{(-1+h)^2 - (-1)^2}{h}$$

$$m = \lim_{h \to 0} \frac{1 - 2h + h^2 - 1}{h}$$

$$m = \lim_{h \to 0} \frac{h^2 - 2h}{h}$$

$$m = \lim_{h \to 0} \frac{h(h-2)}{h}$$

$$m = \lim_{h \to 0} (h-2)$$

$m = -2$

Equation of tangent line:

$m = -2, \ (-1, 1)$

$y = mx + b$

$1 = -2(-1) + b$

$1 = 2 + b$

$-1 = b$

$y = -2x - 1$

Check if tangent line passes (3, -7):

$y = -2(3) - 1$

$y = -6 - 1$

$y = -7$

Therefore Larry does run over the cat.

13. Find the point(s) in exact value where the functions $y = 2x^3 + 10x - 1$ and $y = -\frac{4}{x}$ contain the same slope.

*Use First Principles

$y = 2x^3 + 10x - 1 \rightarrow y' = 6x^2 + 10$

$y = -\frac{4}{x} = -4x^{-1} \rightarrow y' = \frac{4}{x^2}$

$$\Rightarrow$$

$$6x^2 + 10 = \frac{4}{x^2}$$

$$6x^4 + 10x^2 - 4 = 0$$

$$2\left(3x^4 + 5x^2 - 2\right) = 0$$

$$\left(3x^2 - 1\right)\left(x^2 + 2\right) = 0$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}$$

$$y = \mp 4\sqrt{3}$$

no real roots

$\therefore$ Points are at $\left(\frac{1}{\sqrt{3}}, -4\sqrt{3}\right)$ and $\left(-\frac{1}{\sqrt{3}}, 4\sqrt{3}\right)$.

14. Determine the values of b and c in the function $f(x) = x^2 - 3bx + (c+2)$, if $f(x)$ has an x-intercept at $x = 1$ and a $f'(3) = 0$.

*Use First Principles (see below)

$f(x) = x^2 - 3bx + (c+2) \rightarrow f'(x) = 2x - 3b$

$f'(3) = 6 - 3b = 0 \rightarrow b = 2$

$f(1) = 0 \rightarrow 1^2 - 3(2)(1) + (c+2) = 0$

$1 - 6 + c + 2 = 0$

$c = 3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - 3b(x+h) + (c+2) - x^2 + 3bx - (c+2)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3bx - 3bh + (c+2) - x^2 + 3bx - (c+2)}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 - 3bh}{h}$$

$$= \lim_{h\to 0} \frac{h(2x + h - 3b)}{h}$$

$$= 2x - 3b$$