Unit 4: Curve Sketching and Optimization Course Pack

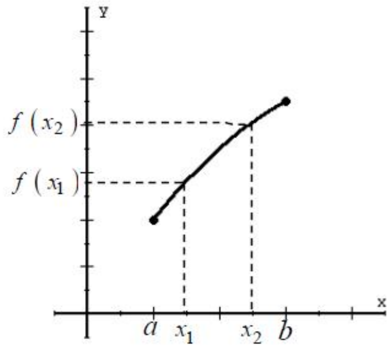
Increasing and Decreasing Functions

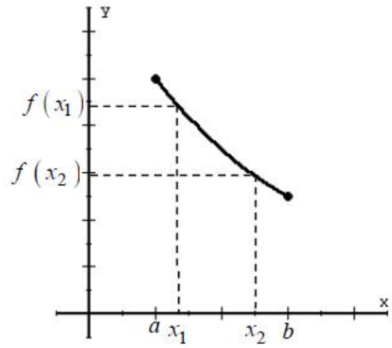
The concepts of increasing and decreasing are closely linked to **intervals** or subsets of a function's domain**.**

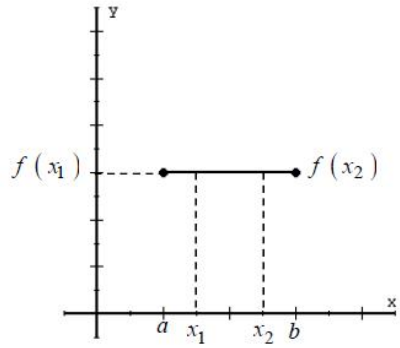
Suppose S is an interval in the domain of $f(x)$, so $f(x)$ is defined for all x in S.

$f(x)$ is **increasing** on S $\Leftrightarrow$ $f(a) \leq f(b)$ for all $a, b \in S$ such that $a < b$

$f(x)$ is **decreasing** on S $\Leftrightarrow$ $f(a) \geq f(b)$ for all $a, b \in S$ such that $a < b$







Test for Increasing and Decreasing Functions

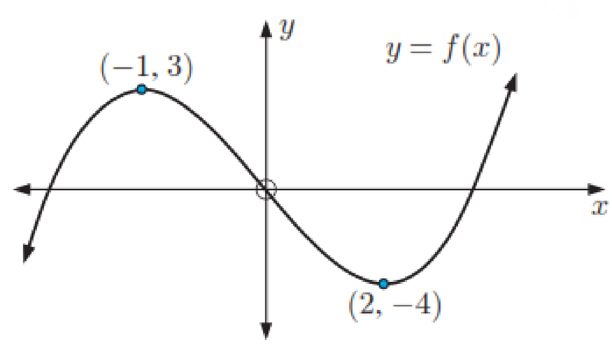
Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) .

- If $f'(x) > 0$ for all $x \in (a,b)$, then f is **increasing** on $[a,b]$
- If $f'(x) < 0$ for all $x \in (a,b)$, then f is **decreasing** on $[a,b]$.
- If $f'(x) = 0$ for all $x \in (a,b)$, then f is **constant** on $[a,b]$.

Example 1: Find intervals where $f(x)$ is:

(a) increasing:___________________________

(b) decreasing: __________________________



At what values of x can the graph of a function change its increasing/decreasing/constant status?

The graph of a continuous function can only change its increasing/decreasing/constant status at a
___________________________________.

Example 3: Find the intervals where the following functions are increasing or decreasing:

a) $f(x) = x^3 - \frac{3}{2}x^2$

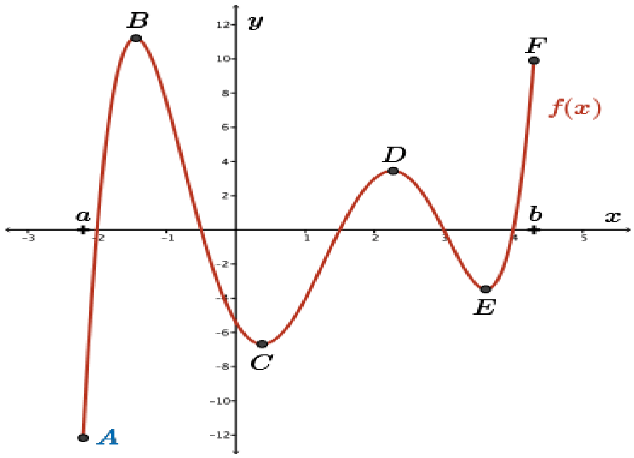
b) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

c) $f(x) = \frac{2x-3}{x^2+2x-3}$

Extrema on an Interval

Extreme Values

Consider the following graph of $y = f(x)$ with domain restricted to a closed interval, $[a, b]$. The high and low points within the curve have been labeled along with the graph's boundary end points. The highest point on the graph is point B. Therefore, point B is known as the **absolute maximum** of $f(x)$ on the interval, $[a, b]$. The lowest point on the graph is point A. Therefore, point A is known as the **absolute minimum** of $f(x)$ on $[a, b]$.



In this case, point A is a **boundary point** of the closed interval, $[a, b]$, specifically at $x = a$.

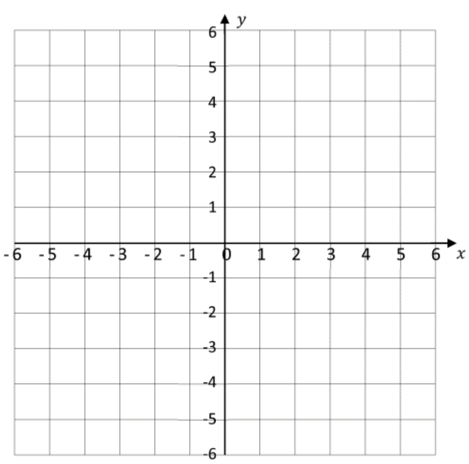
Definitions

A function, *f*, has an absolute maximum at *c* if $f(c) \geq f(x)$ for all *x* in the domain of *f*.

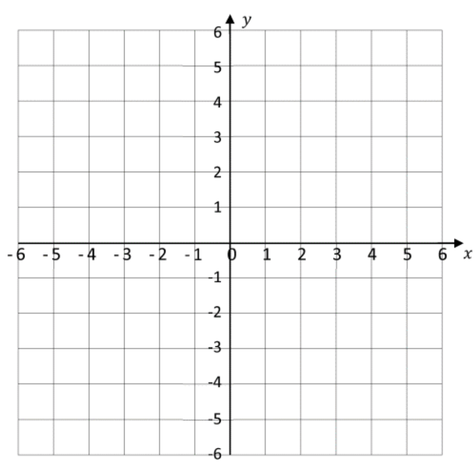
A function, *f*, has an absolute minimum at *c* if $f(c) \leq f(x)$ for all *x* in the domain of *f*.

Example 1: Sketch the following functions on the given interval and determine any extrema, if they exist.

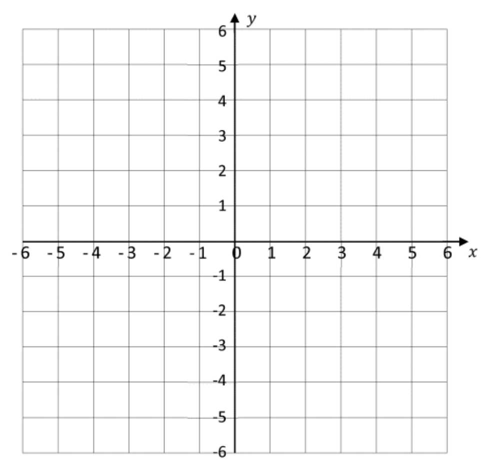
a) $f(x) = x^2 + 1$ on $[-1, 2]$



b) $f(x) = \dfrac{1}{x^2}$ on $[-2, 2]$



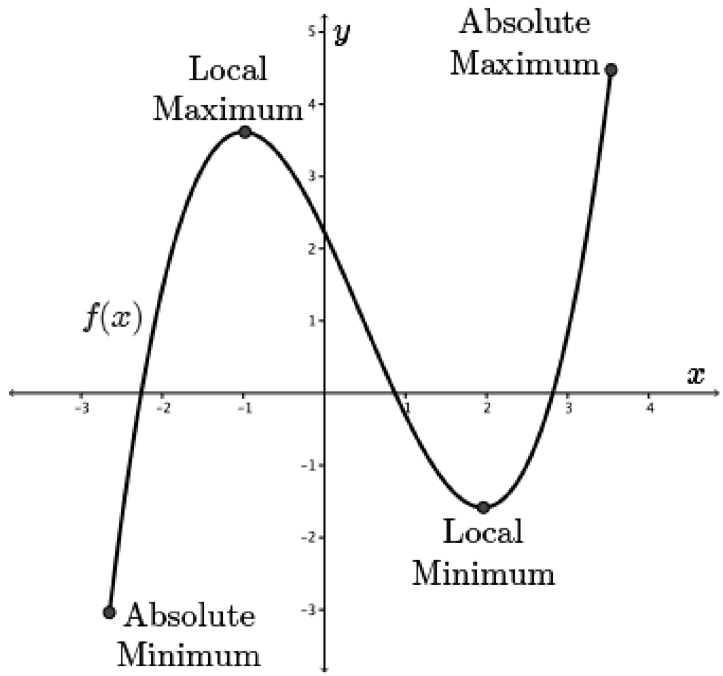
c) $f(x) = 3$ on $[-3, 4]$



What other type of extrema are there?

Definition of Local Maximum and Minimum Values

1. Function, f, has a **local maximum** (or relative maximum) at c if $f(c) \geq f(x)$ for all x sufficiently close to c.
2. Function, f, has a **local minimum** (or relative minimum) at c if $f(c) \leq f(x)$ for all x sufficiently close to c.



Example 2:

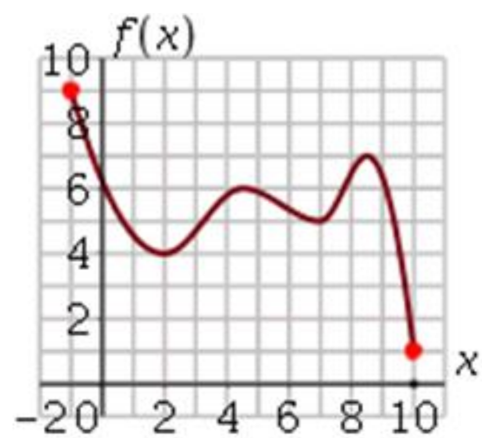
The graph of $f(x)$ is given below. Identify the extrema, both relative and absolute, on the interval [-1, 10] .

Local maximum value(s): ____________________

Local minimum value(s): ____________________

Absolute maximum value: ____________________

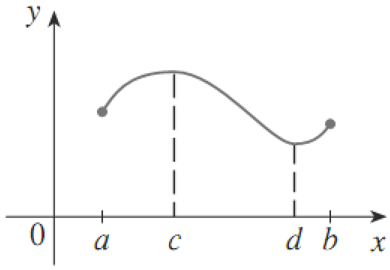
Absolute minimum value: ____________________

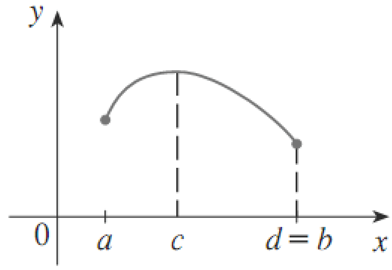


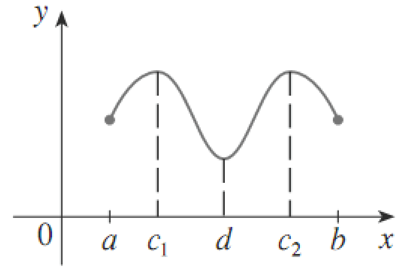
Under what conditions will a maximum or a minimum occur?

The Extreme Value Theorem (EVT)

A continuous function, $f(x)$, defined on a **closed, bounded** interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval.







Definition of Critical Points

The **critical number** of a function is a value c in the domain of the function for which either $f'(c) = 0$ or $f'(c)$ does not exist. If c is a critical number, the point $(c, f(c))$ is a **critical point**.

Example 3: Find all the critical point(s) of the following functions.

a) $f(x) = 6x^5 + 33x^4 - 30x^3 + 100$

b) $f(x) = \dfrac{x^2 + 1}{x^2 - x - 6}$.

Special Note on Fermat's Theorem and Critical Numbers

Fermat's Theorem

Suppose that $f(c)$ is a local extremum. Then c must be a critical number of f.

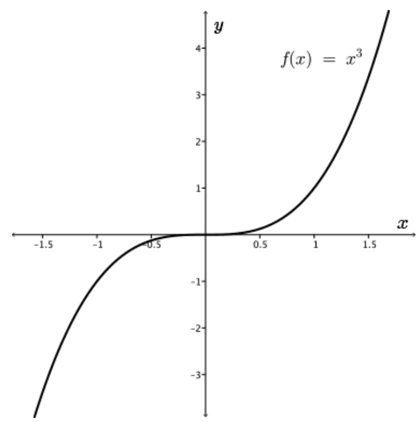
Fermat's Corollary

Suppose that f is continuous on the closed interval $[a, b]$, then the absolute extrema of f must occur at an endpoint, a or b, or at a critical number.

Pierre de Fermat
(1601–1665)

Fermat's theorem is true when it reads forward, but not necessarily true when read backwards. That is, not all values of c that have $f'(c)=0$ are local maximums or minimums.

Example: Consider the function $f(x) = x^3$.



Steps to Finding Extreme Values

Step 1. Identify all of the critical points within the closed interval by finding the values of c where $f'(c) = 0$ and where $f'(c)$ does not exist. Then, find $f(c)$ for each critical number.

Step 2. Find the values of f at the boundary points of the closed interval, $[a, b]$. In other words, find $f(a)$ and $f(b)$.

Step 3. Examine the values of f resulting from step 1 and step 2 and determine which value is the greatest (absolute maximum) and which value is the least (absolute minimum).

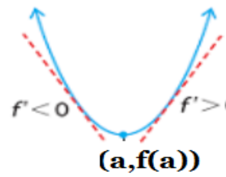
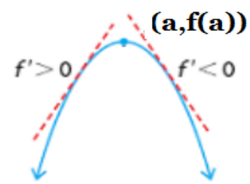
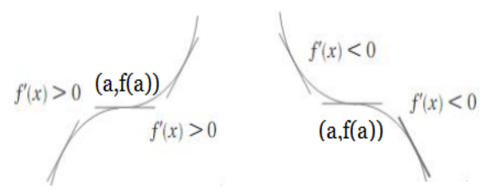
Example 4: Find the absolute extrema of $f(x) = 3x^4 - 12x^3$ on the interval $[-1, 2]$.

Example 5: Determine if the EVT applies. If so, find the absolute extrema of $f(x) = 2x - 3x^{\frac{2}{3}}$ on the interval $[-8, 1]$.

The First Derivative Test

For a continuous function, knowing where the sign of the derivative changes, lends great insight into the existence of any r**elative maxima** or r**elative minima**.

The First Derivative Test (for Relative Extrema)

Let $x = a$ be a critical value of a continuous function, f.	
1. If the sign of $f'(x)$ changes from negative to positive at $x = a$, then f has a **relative minimum** at $x = a$	
2. If the sign of $f'(x)$ changes from positive to negative at $x = a$, then f has a **relative maximum** at $x = a$	
3. If $f'(x)$ is positive on both sides of $x = a$ or negative on both sides of $x = a$, then is neither a relative maximum nor a relative maximum.	

Example 4: Find the local extrema point(s) of the function $f(x) = x^3 - 3x + 2$.

Example 5: Find the relative extrema point(s) of $f(x) = -\frac{x^4 + 1}{x^2}$.

Practice:

1. For $f(x) = x^3 - kx$ where k ∈ R, find the values of k such that f has

   a) no critical numbers b) one critical number c) two critical numbers

2. Find values of a, b, c, and d such that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at (2,4) and a local minimum at (0, 0).

3. For $f(x) = x^2 + px + q$, find the values of p and q such that $f(1) = 5$ is an extremum of f on the interval [0, 2]. Is this extremum a maximum value or a minimum value?

4. For what value of x does the ***derivative*** of $f(x) = \frac{x^4}{3} - \frac{x^5}{5}$ attain its maximum value?

5. Find the **absolute extreme** values for $f(x) = 3x^{\frac{2}{3}}\left(\frac{1}{8}x^2 - \frac{1}{5}x - 1\right)$, $-2 \le x \le 2$.

6. Find the x-coordinate of the point that the function f(x) given by $f(x) = 9x^{\frac{2}{3}} + 3x - 6$ has a relative minimum.

7. Find the *x*-coordinates of the relative extrema of the following functions:

   a) $T(k) = \sqrt[3]{k^2}(2k - 1)$ b) $J(k) = \sqrt[3]{k}(2k - 1)$.

8. Find the intervals where the following functions are increasing or decreasing:

   a) $f(x) = x - 2\sqrt{x}$ b) $f(x) = x^2 + \frac{4}{x-1}$

   c) $f(x) = \frac{x-2}{(x+1)^2}$ d) $f(x) = 3x^{\frac{2}{3}}\left(\frac{1}{8}x^2 - \frac{1}{2}\right)$

Curve Sketching Warm Up

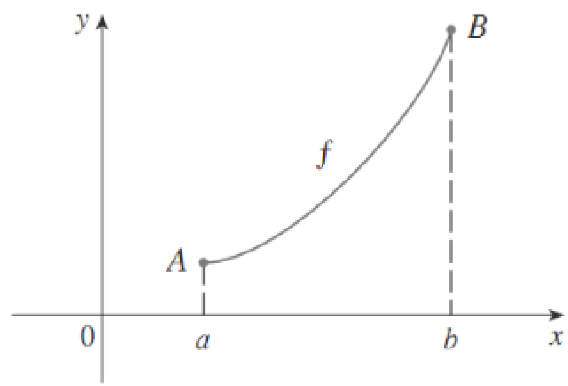
1. Find the absolute maximum and absolute minimum for $f(x) = x^3 + 2x^2 + x - 1$ on $x \in [-1, 1]$.

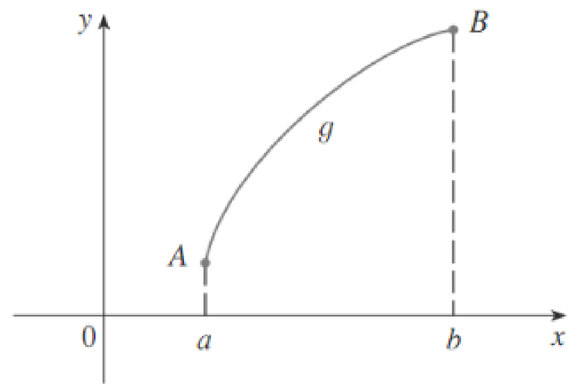
2. Sketch the following functions by completing a full analysis:
   a) $f(x) = x^3 - 9x^2 + 24x - 10$
   b) $y = 3x^{\frac{2}{3}} - 2x$

Concavity and the Second Derivative Test

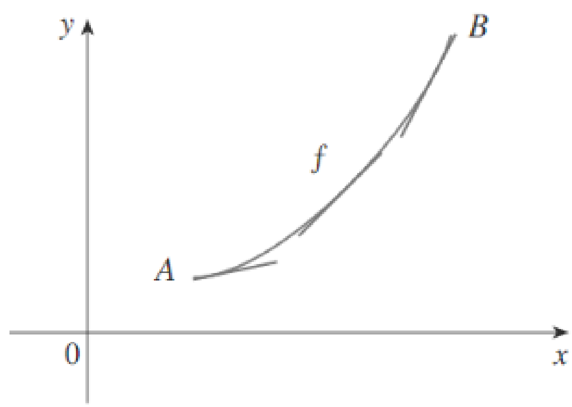
If we know that a function has a positive derivative over an interval, we know the graph of the function is increasing on that interval, but HOW is it increasing? At a constant rate? At an increasing rate? At a decreasing rate?

The two functions below both increase, but they bend differently, and therefore, have different curvature. The function on the left is increasing at an **increasing** rate and the function on the right is increasing at a **decreasing** rate.

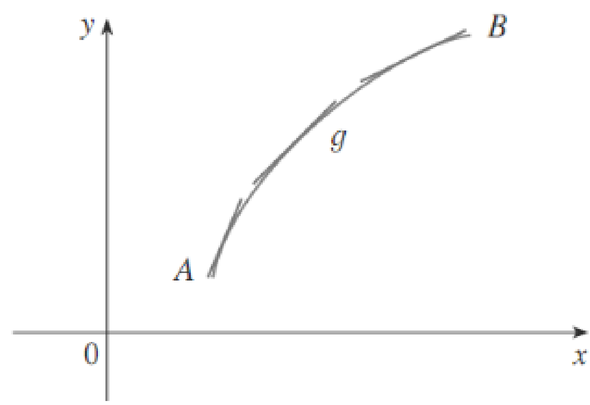




If we analyze the tangent lines in each of these cases at several points, we can begin to talk about how the **slopes**, and not just the y-values are changing.



Concave Up



Concave Down

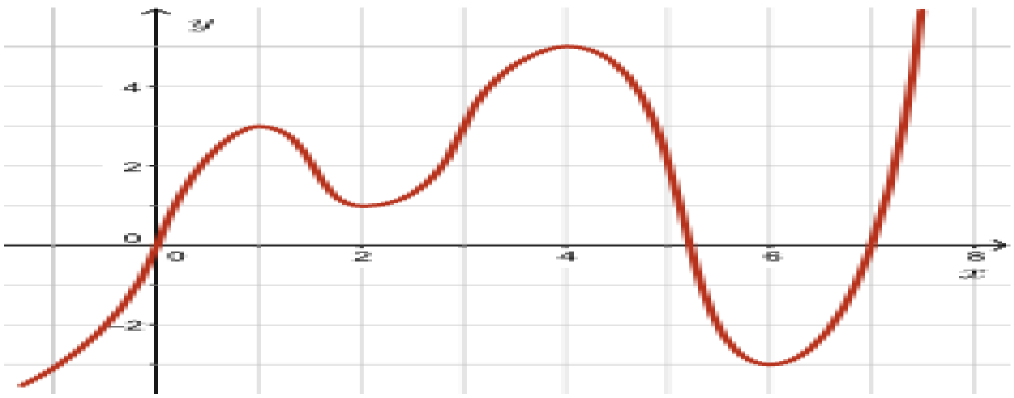
In the graph on the left, the tangent lines are **below** the curve and are increasing from left to right. In this case, we say the graph is **concave up** (like a cup). In this case, the secant lines are above the curve. If the slopes of f are increasing, the second derivative, $f''(x)$, is positive.

In the graph on the right, the tangent lines are **abov**e the curve and are decreasing from left to right. In this case, we say that graph is **concave down** (like a frown). In this case, the secant lines are below the curve. If the slopes of f are decreasing, the second derivative, $f''(x)$, is negative.

Concavity Test

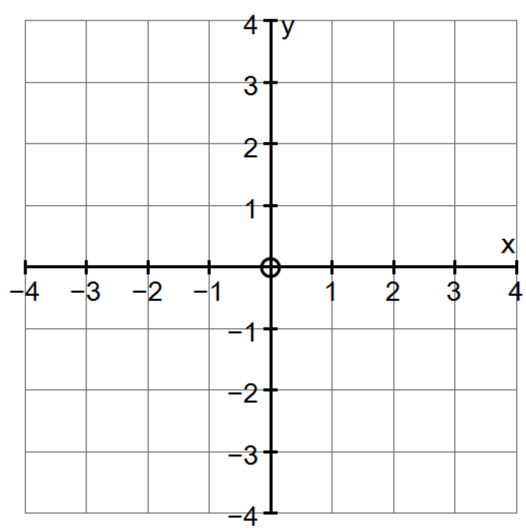
1) If $f''(x) > 0$ for all x in an interval, then $f(x)$ is **concave up** (like a cup) on that interval.
2) If $f''(x) < 0$ for all x in an interval, then $f(x)$ is **concave down** (like a frown) on that interval.

Example 1: In the graph below, list the open intervals on which the graph of the function is concave up (CU) and concave down (CD).



Example 2:
Sketch a possible graph of a function f(x) that satisfies the following conditions:

- $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$
- $f''(x) > 0$ for $x < -2$ *or* $x > 2$ and $f''(x) < 0$ for $-2 < x < 2$
- $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to \infty} f(x) = 0$



Point of Inflection

A point $(c\ , f(c))$ on a curve $y = f(x)$ is called an **inflection point** if the graph of f changes from concave up to concave down OR from concave down to concave up at $(c\ , f(c))$.

That is to say if $f\ ''(c) = 0$ or $f\ ''(c)$ does not exist and the sign of $f''(x)$ changes from positive to negative or from negative to positive at $x = c$, then $x = c$ is an inflection point of $f(x)$.

Example 3: Determine the open intervals on which the graphs of the following functions are concave up or concave down and find the inflection point(s), if any.

a) $y = 3 + x^4$

b) $y = 6\left(x^2 + 3\right)^{-1}$

c) $f(x) = \frac{x-1}{x+4}$

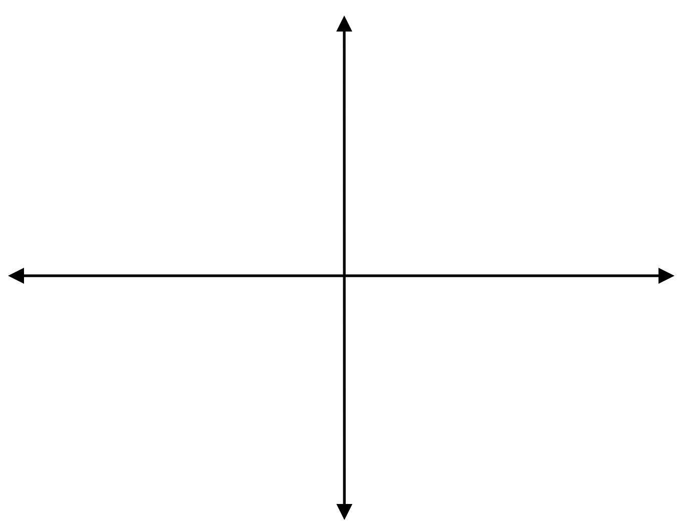
d) $f(x) = x^4 - 4x^3$

The Second Derivative Test (for Relative Extrema)

Let f be a function such that $x = c$ is a critical value of f such that $'(c) = 0$. If $f"(x)$ exists on an open interval containing $x = c$, then:

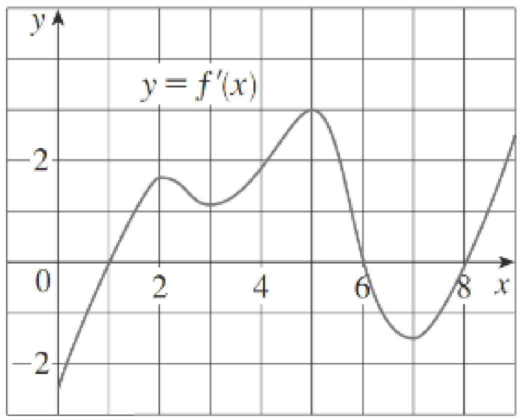
1) If $"(c) > 0$, then $(c, f(c))$ is a relative **minimum**.
2) If $"(c) < 0$, then $(c, f(c))$ is a relative **maximum**.
3) If $f"(c) = 0$, then the test fails and the First Derivative Test must be used.

Example 4: Find the coordinates of the relative extrema for $f(x) = -3x^5 + 5x^3$ using the 2nd Derivative Test, if possible, and locate any inflection point(s). Sketch the graph.

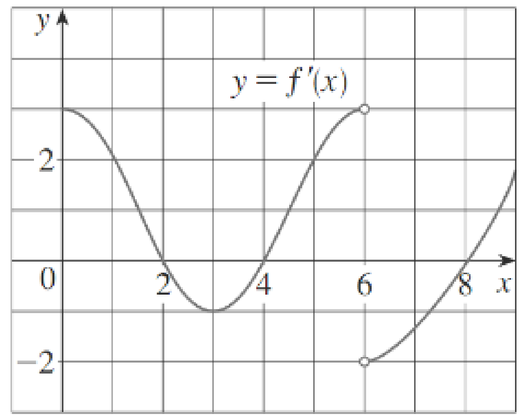


Example 5: The graph of the derivative, $f'(x)$, of two continuous functions, $f(x)$, on the interval [0, 9] are shown below.

I.



II.



Answer the following questions for each.

(a) On what open intervals is $f(x)$ increasing or decreasing?

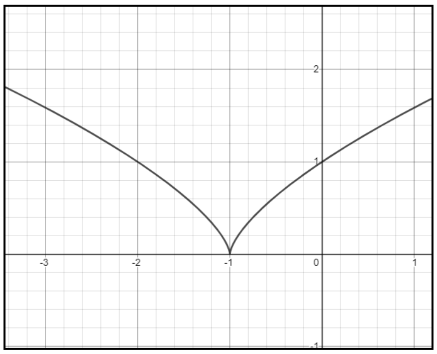
(b) At what value(s) of x does $f(x)$ have a local maximum or local minimum?

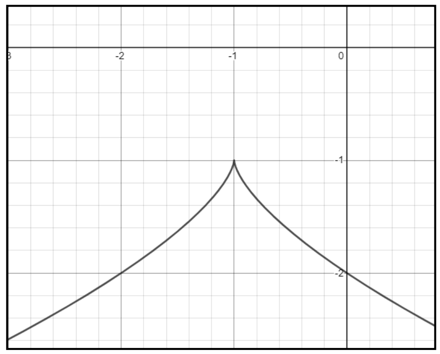
(c) On what intervals is $f(x)$ concave up or concave down?

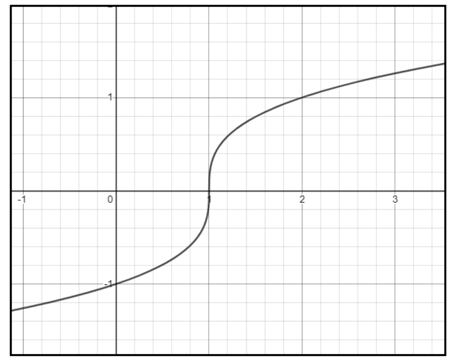
(d) Assuming that $f(0) = 0$, sketch a graph of $f(x)$.

Cusps

A cusp is a pointy bend on a graph where no tangent line exists. A cusp is a good example of the situation where a function can have a maximum or minimum at a point where the derivative of the function does not exist. To spot a cusp, determine where the derivative does not exist and test the concavity on either side of that critical number. If the concavity remains the same, you have found a cusp. If the concavity changes, you have found a vertical tangent.







Example 1: Determine if the following functions have a cusp. If yes, determine the point.

a) $f(x) = (x+6)^{\frac{1}{2}} - 2$

b) $f(x) = \sqrt[3]{x^2}$

4-4 Warm Up

1. The function $f(x) = 2ax^3 + 3x^2 + bx - 3$ has a local minimum at $x = -1$ and a point of inflection at $x = 1$. Determine the values of a and b.

2. Use the 2nd Derivative Test to determine the coordinates of the local extrema of the function $f(x) = 3x^4 - x^3 - 6x^2 + 3x - 11$.

Solving Optimization Problems

Strategies:

1. Whenever possible, draw a diagram, labeling the given and required quantities. Identify the variables.
2. Determine a function that represents the quantity to be optimized. Be sure that this function only depends on one variable. (If there are two variables, look for information to create a second equation.)
3. Differentiate and find the critical number (i.e. find the maximum or minimum function value) by setting the first derivative to zero.
4. Determine the optimal value and check constraints.
5. Answer the question in your conclusion.

Number Type

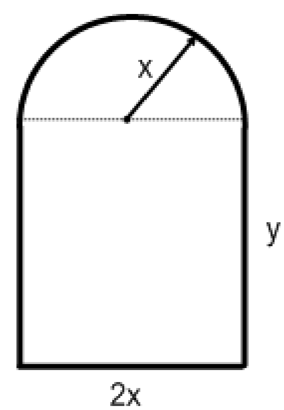
1. Express 24 as the sum of 2 positive numbers such that the product of one by the square of the other is a maximum.

Optimization: Area and Perimeter

1. Three hundred meters of fencing is available to enclose a rectangular field and divide it into two smaller equal plots. What should the dimensions of the smaller plots be for the area to be a maximum?

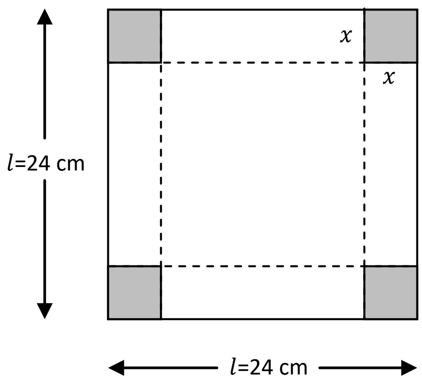
2. A Norman window consists of a rectangle surmounted by a semi-circle. If the perimeter of the window is 8m, find the width of the window that will admit the greatest amount of light.

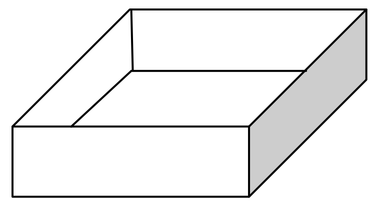
   Let 2x represent the width and let y represent the height.



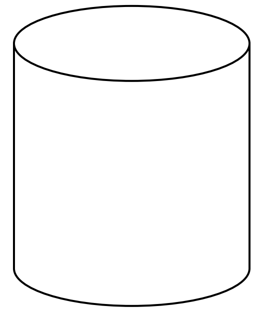
Optimization: Surface Area and Volume

1. A piece of sheet metal 24 cm by 24 cm is to be used to make a rectangular box with an open top. The box is to be made by cutting equal squares from the corners and folding up the flap to make the sides. Find the dimensions that will give the box with the largest volume.



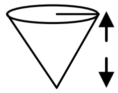
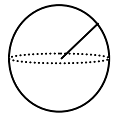
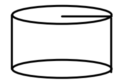
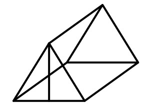
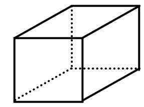


2. A manufacturer wishes to produce cylindrical fruit juice cans with a capacity of 250 mL. What dimensions will minimize the amount of material required for a can? (1 mL=1 cm^3)



OPTIMIZING INSCRIBED SHAPES

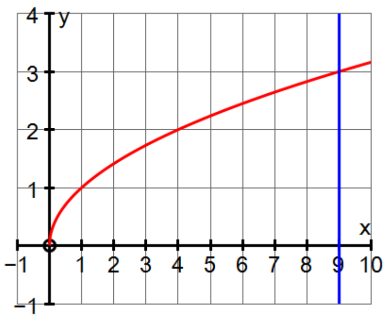
Some important formulae:

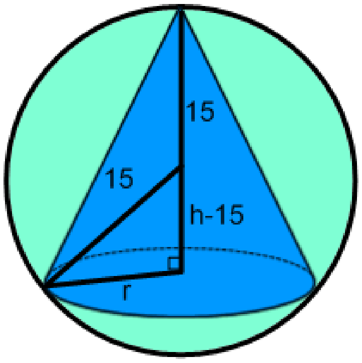
$V_{cone} = \frac{1}{3}\pi r^2 h$ $\quad V_{sphere} = \frac{4}{3}\pi r^3$ $\quad V_{cylinder} = \pi r^2 h$ $\quad v_{prism} = A_{base} \times h$

$SA_{sphere} = 4\pi r^2$ $\quad 1\ cm^3 = 1\ mL$

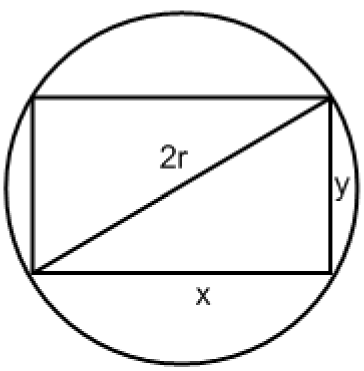
1. Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bound by the *x*-axis, $x = 9$ and the graph $= \sqrt{x}$.



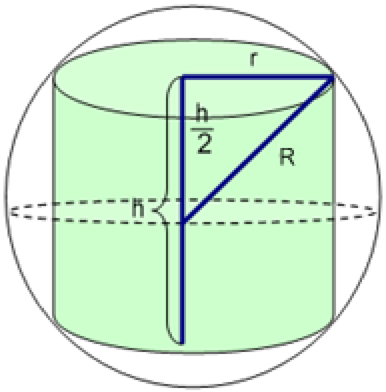
2. Determine the dimensions of the largest cone which can be inscribed in a sphere, where the diameter of the sphere is fixed at 30 cm.



3. Find the dimensions of the largest rectangle which can be inscribed in a circle, where the area of the circle is fixed at $100\pi\ cm^2$.



4. Find the dimensions of the cylinder of greatest volume which can be inscribed in a sphere, where the volume of the sphere is fixed at $\frac{108\pi}{3}\ m^3$.



<u>OPTIMIZING AREA & VOLUME</u>

1. Determine the dimensions of the largest cone which can be inscribed in a sphere, where the diameter of the sphere is fixed at 30 cm. [Answer: $r = 10\sqrt{2}\ cm$, $h = 20\ cm$]

2. Find the dimensions of the largest rectangle, by perimeter, which can be inscribed in a circle, where the area of the circle is fixed at $100\pi\ cm^2$. [Answer: $10\sqrt{2}\ cm \times 10\sqrt{2}\ cm$]

3. Find the dimensions of the cylinder of greatest volume which can be inscribed in a sphere, where the volume of the sphere is fixed at $\frac{108\pi}{3}\ m^3$. [Answer: $r = \sqrt{6}\ cm$, $h = 2\sqrt{3}\ cm$]

4. Find the dimensions of the rectangle of maximum perimeter that can be inscribed in a circle of radius 4 cm. [Answer: $4\sqrt{2}\ \ cm\ by\ 4\sqrt{2}\ cm$]

5. Find the dimensions of the rectangle of maximum area that can be inscribed in an isosceles triangle with base 40 *cm* and height 30 *cm*. [Answer: 15 cm by 20 cm]

6. Find the dimensions of the cylinder of maximum volume that can be inscribed in a cone with a diameter of 40 *cm* and a height of 30 *cm*. [Answer: $h = 10cm, r = \frac{40}{3}cm$]

7. Find the height and radius of the cylinder of greatest volume that can be inscribed in a sphere of radius ***R*** units. [Answer: $h = \frac{2\sqrt{3}R}{3}, r = \frac{\sqrt{6}R}{3}$]

8. A rectangular box is made from a piece of cardboard which measures 48 cm by 18 cm by cutting equal squares from each corner and turning up the sides. Find the maximum volume of such a box if:
   a) the height of the box must be at most 3 cm. [Answer: 1512 cm^3]
   b) the length and width of the base must be at least 10 cm. [Answer: 1600 cm^3]

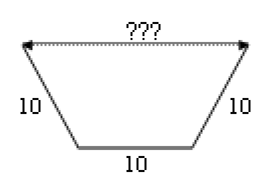
9. A piece of paper for a poster has an area of $1\ m^2$. The margins at the top and bottom are 8 cm and at the sides are 6 cm. What are the dimensions of the sheet of paper which will maximize the printed area of the page? [Answer: $\frac{2\sqrt{3}}{3}m\ by\ \frac{2\sqrt{3}}{3}m$]

10. The area of a rectangle is $64\ cm^2$. Find the dimensions of the rectangle of minimum perimeter. What is the minimum perimeter? [Answer: 8 cm by 8cm; 32 cm]

11. A piece of wire 100 cm long is divided into two pieces. One piece is used to form a circle and the other a square. Find the lengths of wire cut so that the combined area of circle and square is a minimum. [Answer: 43.99 cm and 56.01 cm]

12. In above question, into what lengths should the wire be divided to give a combined area as large as possible? [Answer: one piece of wire 100 cm long forming a circle]

13. An eaves trough has a cross section that forms an isosceles trapezoid. If the two legs and the short base of the trapezoid are each 10 cm, find the distance across the top of the trapezoid that will maximize the area of the trapezoid and thus the carrying capacity of the eaves trough. [Answer: base of 20 cm]



Warm Up-Optimizing Volumes and Surface Areas

1. A rectangular open-topped box is to be made from a piece of material 18 cm by 48 cm by cutting a square from each corner and turning up the sides. What size squares must be removed to maximize the capacity of the box?

2. If 3600 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.

3. The volume of a square-based rectangular cardboard box is to be 1000 cm^3. Find the dimensions so that the quantity of material used to manufacture all 6 faces is a minimum. Assume that there will be no waste material. The machinery available cannot fabricate material smaller in length than 2 cm.

Optimization: Revenue and Cost

Revenue Type

1. A railway company offers excursions at $120 per person for tour groups of up to 100 people. If the size of the group is greater than 100, the company will reduce the price of every ticket by 50 cents for each person in excess of 100. What size tour group would produce the greatest revenue?

Cost Type

2. An open topped storage box with a square base is to have a capacity of 5 m^3. Material for the sides costs \$1.60/m^2 while that for the bottom costs \$2/m^2. Find the dimension that will minimize the cost of the material. What is the minimum cost?

OPTIMIZATION PRACTICE – BUSINESS

1. A car rental agency has 200 cars. The owner finds that at a price of $36 per day he can rent all the cars. For each $2 increase in price, the demand is less and 5 fewer, cars are rented. What price will maximize the total revenue?

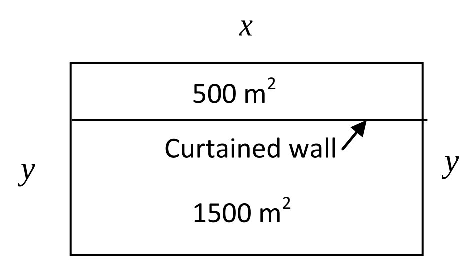
2. The current ticket price at a local theatre is $4 and the theatre attracts an average of 250 customers per show. Every $0.20 increase in ticket price reduces the average attendance by 10 customers. Find the ticket price that will maximize the revenue.

3. A variety store can sell 500 yo-yos for $1 each. For each cent the store lowers the price, it can sell 20 more yo-yos. For what price should it sell the yo-yos to make maximum revenue?

4. A retailer of electrical appliances can sell 200 refrigerators at $250 each. For each reduction of $10 in price 10 more refrigerators per month are sold. But reductions of less than $10 per unit have no effect on sales. What selling price would produce the maximum revenue per month and how many refrigerators would be sold at this price?

5. A real estate firm owns 250 apartments that can be rented out at $460 per month each .For each $5 per month increase in rent there are two vacancies created that cannot be filled .What should the monthly rent be to maximize the revenue? What is the maximum revenue?

6. A rectangular building is to be constructed with a curtain wall that will be parallel to the front, and will divide the space into a sales area of 1500 m^2, and a storage area of 500 m^2. The outside wall on both sides and across the back will be constructed of concrete block at a cost of $100 per linear meter. The front wall, which will be comprised primarily of glass, will cost $345 per linear meter, while the curtained wall will cost $45 per linear meter. Find the dimension of the building that will give the required floor areas at the lowest cost for all of the walls.



Answers:

1. $\$58/car$

2. $\$4.50/ticket$

3. $\$0.625/yo-yo$

4. 230 units at $220 or

   220 units at $230

5. $542.50/unit for $117722.50 in revenue

6. dimensions $28.57m \times 70\ m$

Warm Up-Optimizing Revenue and Costs

1. A real estate firm owns 250 apartments that can be rented out at $460 per month each. For each $5 per month increase in rent, there are 2 vacancies created that cannot be filled. What should the monthly rent be to maximize the revenue? What is the maximum revenue?

2. A car rental agency has 200 cars. The owner finds that at a price of $36 per day, he can rent out all the cars. For each $2 increase in price, 5 fewer cars are rented. What price will maximize the total revenue?

3. A supermarket is designed to have a rectangular floor area of 3750 m^2 with 3 walls made of cement blocks and one wall made of glass. In order to conform to the building code, the length of the glass wall must not exceed 60 m but must not be less than 30 m. The cost per linear meter of constructing a glass wall is twice that of constructing a cement wall. Find the dimensions of the floor area that will minimize the cost of building the walls.

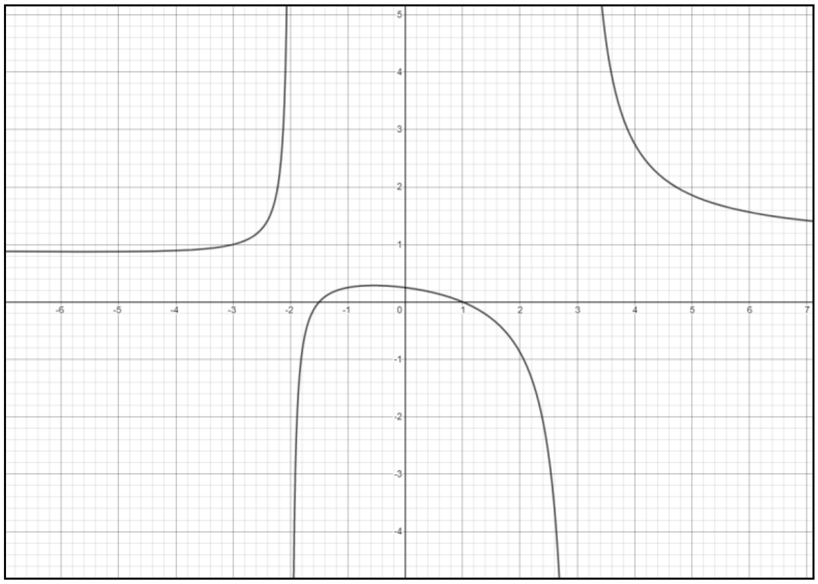
4-7 Warm Up

1. Find and classify all the critical points
   a) $p(x) = 2x^3 - 4x^2 - 8x + 6$
   b) $h(t) = -4.9t^2 + 33.8t + 1.5$

2. Given the function: $g(x) = 1 + 3x^2 - 2x^3$, find all critical numbers, intervals of increase and decrease, and concavity values of the function. Use the information to sketch a graph.

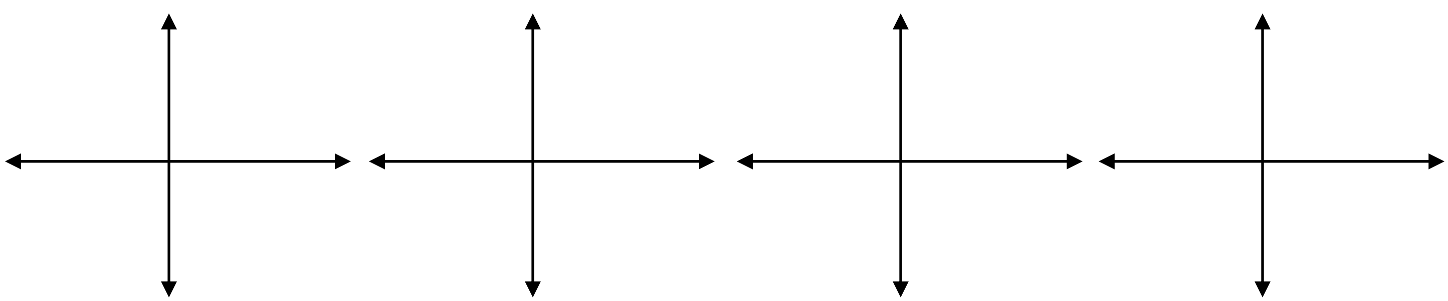
Vertical Asymptotes

Example: Identify the asymptotes in the following graph.



Vertical asymptotes can be found by looking to see where the y-values of a function approaches ∞ or $-\infty$. Horizontal asymptotes can be found by looking to see what happens as x gets bigger and bigger or as x gets smaller and smaller (i.e., as $x \to \infty$ or as $x \to -\infty$).

The graphs of rational functions often contain asymptotes. For example, $f(x) = \frac{2(x^2+1)}{(x-1)(x+2)}$ has two vertical asymptotes at $x = 1$ and $x = -2$. Suppose we wanted to graph this function. Consider all the possible 'behaviours' of the function near these two vertical asymptotes.



We can use one-sided limits to decide whether the function is approaching ∞ or $-\infty$ on either side of a vertical asymptote.

Ex: Sketch a graph of $f(x) = \frac{2(x^2+1)}{(x-1)(x+2)}$ near its vertical asymptotes.

Summary for Vertical Asymptotes

To determine the behavior of a rational function near its vertical asymptotes:

1) Factor and reduce the expression, if possible. (Note: There may be a hole if the factor as a variable.)

2) Locate vertical asymptotes by finding values of x that make the denominator of the simplified function zero.

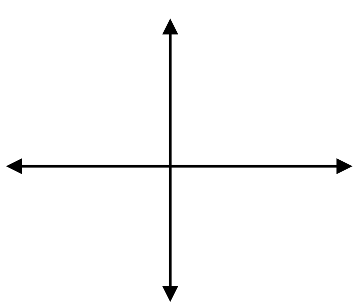
3) If $x = a$ is a vertical asymptote, calculate $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$. If this limit will either be ∞ or $-\infty$. To decide which, consider the sign of each of the factors of $f(x)$.

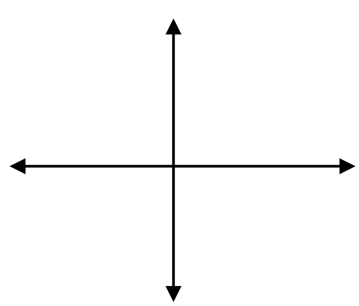
Example: sketch each of the following around the vertical asymptotes.

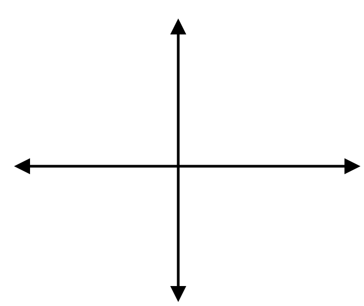
a) $f(x) = \frac{3-2x}{x-3}$

b) $f(x) = \frac{x}{x^2-x-6}$

c) $f(x) = \frac{x-3}{x^2+x-12}$







4-8 Warm Up

1. Find the absolute maximum and minimum values of $f(x) = 2x^3 - 3x^2 - 36x + 62$ on the interval $x \in [-3, 4]$.

2. Sketch a graph that satisfies the following conditions:
   - $f''(x) < 0, x \in (-2,1)$
   - $f''(x) > 0, x \in (-\infty, -2) \cup (1, \infty)$
   - $f(-2) = -3$
   - $f(0) = 0$
   - $f'(-3) = 0$
   - $f'(2) = 0$

3. Sketch the graph of $f(x) = x^{\frac{2}{3}}(x - 5)$

4. Find the equation of the cubic function which has a point of inflection at (0, 2) and a local maximum at (2, 6).

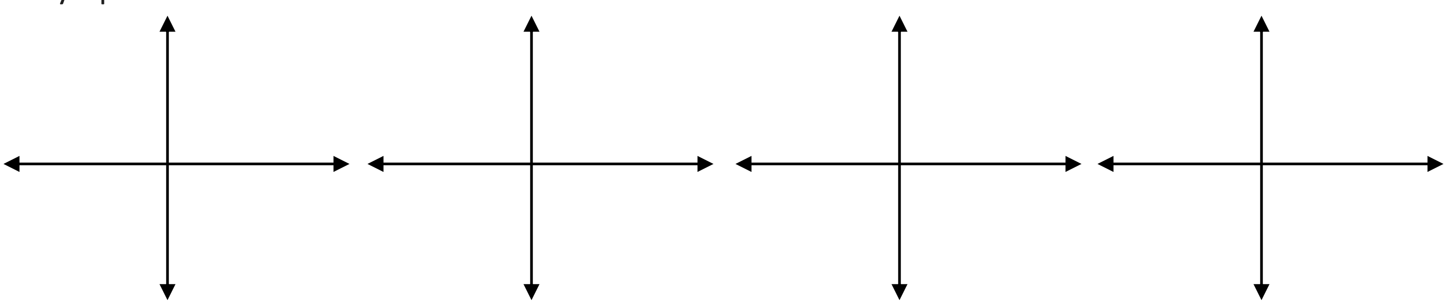
Horizontal Asymptotes

Recall from the previous lesson that horizontal asymptotes can be found by looking to see what happens as x gets bigger and bigger or as x gets smaller and smaller (i.e., as $x \to \infty$ or as $x \to -\infty$).

We can use this idea to find the horizontal asymptotes for the function $f(x) = \frac{2(x^2+1)}{(x-1)(x+2)}$.

First, determine $\lim_{x\to\infty} \frac{2(x^2+1)}{(x-1)(x+2)}$ and $\lim_{x\to-\infty} \frac{2(x^2+1)}{(x-1)(x+2)}$.

Draw diagrams to show the different ways in which our function may be approaching this asymptote.



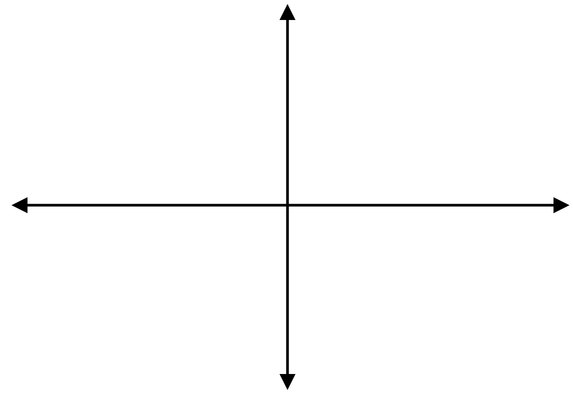
How can we decide which scenario is the correct one? There are two methods.

1) Determine the sign for $\lim_{x\to\infty} f(x) - L$ and $\lim_{x\to-\infty} f(x) - L$, where $y = L$ is the horizontal asymptote.
    - If the limit is **positive**, then $f(x)$ is **above** the horizontal asymptote.
    - If the limit is **negative**, then $f(x)$ is **below** the horizontal asymptote.

OR

2) Calculate $f(1000)$ and $f(-1000)$ to test the behavior of the function at the extremities of the graph.
    - If $f(1000) > L$ or $f(-1000) > L$, the function is approaching the horizontal asymptote from above.
    - If $f(1000) < L$ or $f(-1000) < L$, the function is approaching the asymptote from below.

Example: Determine the end behavior of the function $f(x) = \frac{2(x^2+1)}{(x-1)(x+2)}$ and sketch a possible graph using the information gathered over the last two lessons.



Summary for Horizontal Asymptotes

To determine the behavior of a rational function near its horizontal asymptotes:

1) Locate horizontal asymptotes by calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is the equation of the horizontal asymptote.

2) Calculate $f(1000)$ and $f(-1000)$. If $f(1000) > L$ or $f(-1000) > L$, the function is approaching the horizontal asymptote from above. If $f(1000) < L$ or $f(-1000) < L$, the function is approaching the asymptote from below.

Example: Sketch a possible graph of the function, $(x) = \frac{3-2x}{x-3}$, by analyzing its behavior near its asymptotes.

Curve Sketching Summary

1) Increasing: a function is increasing where $f'(x)$___________
   Decreasing: a function is decreasing where $f'(x)$___________

2) First Derivative Test
   - If $f'(x)$ changes sign from positive to negative at c, then $f(x)$ has a local _____________________ at c.
   - If $f'(x)$ changes sign from negative to positive at c, then $f(x)$ has a local _____________________ at c.

   Draw sketches to illustrate the First Derivative Test:

3) Critical Number: c is a critical number if c is in the domain of the function and either
   i. $f'(c) =$ ______ OR ii) $f'(c) =$________

   If $f'(c) = 0$, what might be happening at c?

   If $f'(c)$ does not exist, what might be happening at c?

4) Vertical Asymptotes
   - simplify the rational expression first, then look at denominator
   - If $x = a$ is a vertical asymptote, calculate ____________and ______________. This limit will either be _________ or __________. To decide which, consider the sign of each of the factors of $f(x)$.

5) Horizontal Asymptotes

- Determine if ____________ or _____________ exists.
- Once a horizontal asymptote has been found, you must determine whether the function is approaching the asymptote from above or below.

6) Points of Inflection and Concavity

Test for Concavity:

- If $f"(x) > 0$ for all x on some interval I, then the graph of f is _______________ on I.
- If $f"(x) < 0$ for all x on some interval I, then the graph of f is _______________ on I.

To Find Possible P. O. I:

- Solve $f"(x) =$_______ and determine where $f"(x)$___________.
- Use an interval chart to determine on which intervals $f"(x)$ is positive or negative.

A point of inflection occurs when…

7) Second Derivative Test:

- If $f'(c) = 0$ and $f"(c) > 0$, then f has a local __________________ at $x = c$.
- If $f'(c) = 0$ and $f"(c) < 0$, then f has a local ___________________ at $x = c$.

Draw diagrams to illustrate the Second Derivative Test

A Finalized Algorithm for Curve Sketching

An Algorithm for Curve Sketching

1) Determine the domain of the function. Any value for which the function is undefined creates a discontinuity in the function (either a hole or an asymptote).
2) Determine the y-intercept, and, if easy to do, any x-intercept(s).
3) Determine the behavior of the function near any vertical asymptotes.
4) Determine the behavior of the function near any horizontal asymptotes.
5) Determine the critical points by solving $f'(x) = 0$ and by determining where $f'(x)$ does not exist.
6) Test any critical points by using the First Derivative Test or by using the Second Derivative Test.
7) Determine the possible Points of Inflection by solving $f''(x) = 0$ and by determining where the $f''(x)$ does not exist. Test the concavity on either side of a possible P. O. I.
8) Combine information from above to produce a sketch of the function.

The steps outlined above are not always necessary or always sufficient to sketch: your intuition and experience with functions can help you out a lot.

EXAMPLE: Use the Algorithm for Curve Sketching to graph the following:

a) $f(x) = \frac{3x^3-2}{x^3+1}$

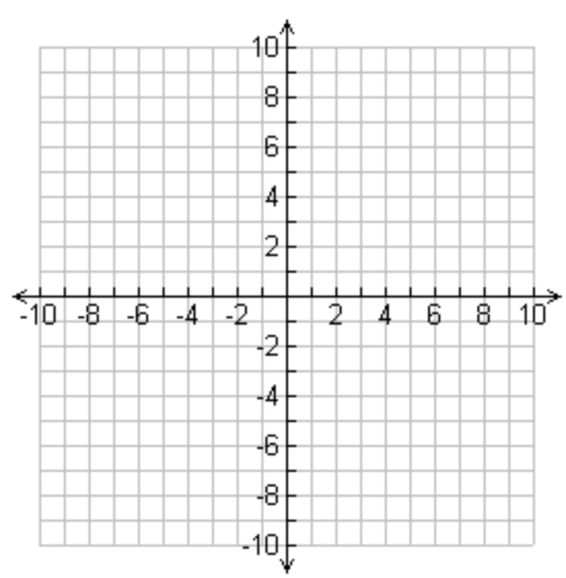
b) $f(x) = \frac{x}{(x-1)^2}$

Curve Sketching Putting It All Together

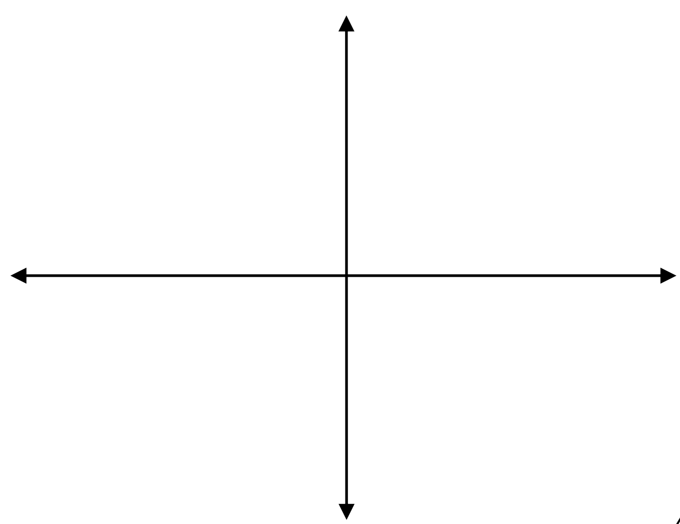
1. Determine the intervals of increase and decrease for the function $f(x) = \frac{1}{4}x^4 - 2x^3$.

2. Determine the relative extrema of the function $f(x) = x^3 - 3x + 2$.

3. Sketch a graph of the function, $f(x) = \frac{x^2-x-2}{x-1}$, by determining:
   a. The domain
   b. x-intercept(s) and y-intercept
   c. Critical points
   d. Vertical and Horizontal Asymptote(s)
   e. Point(s) of inflection



4. Sketch the graph of $f(x) = x^{\frac{2}{3}} - 2x^{\frac{5}{3}}$ by completing a full analysis. Use the following to help you: $f'(x) = \frac{2}{3}\left(\frac{1-5x}{x^{\frac{1}{3}}}\right)$ and $f''(x) = -\frac{2}{9}\left(\frac{1+10x}{x^{\frac{4}{3}}}\right)$.



Review: Maximum and Minimum Problems

1. If the sum of two positive numbers is 28, prove that the square of one added to the cube of the other is at least 640.

2. What is the maximum possible area of a rectangle with a base that lies on the x-axis and two upper vertices that lie on the graph of $y = 4 - x^2$.

3. Find the point on the hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ that is nearest to the point$(6, 0)$.

4. A cylindrical tin can, closed at both ends, of a given capacity, has to be constructed. Show that the amount of tin required will be a minimum when the height is equal to the diameter.

II. Curve Sketching from Given Information

1 .Sketch a graph of a function that satisfies all of the following conditions:

- The domain of the function is the set of all real numbers except $x = 1$ and $x = -1$.
- $\lim_{x\to 1^+} f(x) = -\infty$, $\lim_{x\to 1^-} f(x) = \infty$, $\lim_{x\to -1^+} f(x) = \infty$, $\lim_{x\to -1^-} f(x) = -\infty$
- $\lim_{x\to\infty} f(x) = 0$, $f(x) < 0$ as $x \to \infty$
- $\lim_{x\to-\infty} f(x) = 0$, $f(x) < 0$ as $x \to -\infty$
- The y intercept is 2. There are no x intercepts.

2. Sketch a graph of a polynomial (continuous) function that satisfies all of the following conditions:

- $f'(x) > 0$ for $0 < x < 1$ and $f'(x) < 0$ for $1 < x < \infty$
- $f''(x) > 0$ for $2 < x < \infty$ and $f''(x) < 0$ for $0 < x < 2$
- $\lim_{x\to\infty} f(x) = 0$
- $f(x)$ is an odd function (symmetrical about the origin)

3. Sketch a graph of a (continuous) function that satisfies all of the following conditions:

- $f(0) = f(3) = 2, \quad f(-1) = f(1) = 0$
- $f'(-1) = f'(1) = 0$
- $f'(x) < 0$ for $-\infty < x < -1$ and for $0 < x < 1$
- $f'(x) > 0$ for $-1 < x < 0$ and for $1 < x < \infty$
- $f''(x) > 0$ for $x < 3\,(x \neq 0)$ and $f''(x) < 0$ for $3 < x < \infty$
- $\lim_{x\to\infty} f(x) = 4$ and $\lim_{x\to-\infty} f(x) = \infty$

4. Sketch a graph of a function that satisfies all of the following conditions:

- $f(0) = 1, \quad f(1) = 2$
- $\lim_{x\to 2^-} f(x) = \infty$, $\lim_{x\to -1^+} f(x) = -\infty$

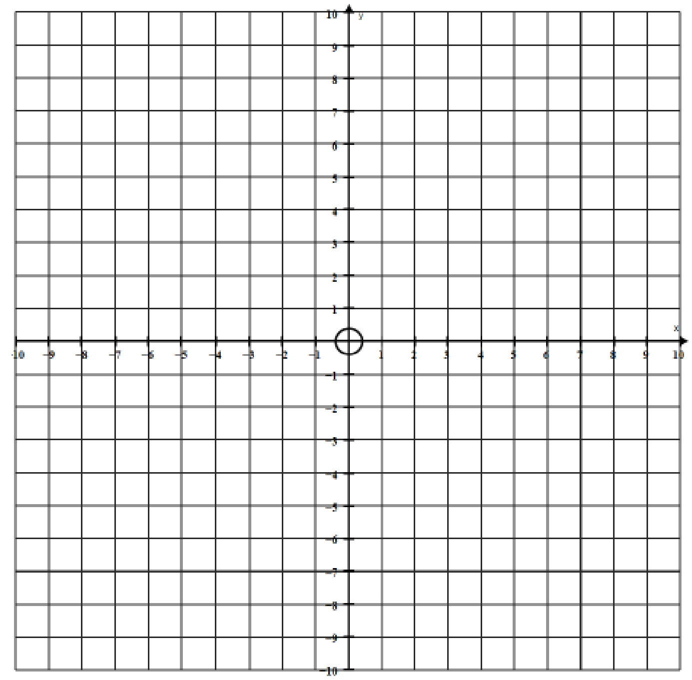
5. Sketch a graph of a polynomial (continuous) function that satisfies all of the following conditions:

- $f(3) = 5, \quad f(7) = -2, \quad f(5) = 2$
- $f'(3) = f'(7) = 0$
- $f'(x) > 0$ for $x < 3$ and $x > 7$, $f'(x) < 0$ for $3 < x < 7$
- $f''(5) = 0$
- $f''(x) > 0$ for $x > 5$ and $f''(x) < 0$ for $x < 5$

6. Graph $f(x)$ if $f'(x) < 0$ when $x < -2$ and when $x > 3$, and $f'(x) > 0$ when $-2 < x < 3$, and $f(-2) = 0$ and $f(3) = 5$.

Warm up: Curve Sketching

1. Sketch a possible graph of the continuous function that satisfies all of the following conditions:



- $f(-7)=5, f(-5)=f(3)=-8, f(-1)=-2$
- $f'(-5)=f'(-1)=0$
- $f'(-7)=f'(3)=DNE$
- $f'(x)<0 \quad for \quad x\in(-\infty,-7)\cup(-7,-5)\cup(-1,3)$
- $f'(x)>0 \quad for \quad x\in(-5,-1)\cup(3,\infty)$
- $f''(-3)=0$
- $f''(-7)=f''(3)=DNE$
- $f''(x)<0 \quad for \quad x\in(-\infty,-7)\cup(-3,3)\cup(3,\infty)$
- $f''(x)>0 \quad for \quad x\in(-7,-3)$
- $\lim_{x\to-\infty} f(x)=\infty \quad and \quad \lim_{x\to\infty} f(x)=5$

2. Given the function $f(x)=\dfrac{ax+b}{x^2-c}$ and that it has the following properties:

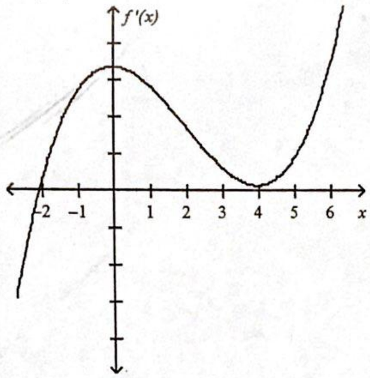
- the graph of $f(x)$ is symmetric with respect to the y-axis
- $\lim_{x\to 2^+} f(x)=+\infty$
- $f'(1)=-2$

(a) Determine the values of a, b & c.

(b) Determine the equation for all asymptotes of the graph of $f(x)$.

Multiple Choice: Write the CAPITAL LETTER corresponding to the correct answer on the line provided

______1. Below is the graph of $\mathbf{f'(x)}$. For what value(s) of x does $\mathbf{f(x)}$ have a local minimum?



A. $x=-2$ B. $x=0$ C. $x=4$ D. None

______2. Let $f(x)$ be a continuous function. If $f'(x)>0$ for $x>c$ and $f'(x)<0$ for $x<c$ then what type of critical point is $(c,f(c))$?

A. Local maximum
B. Local minimum
C. Neither a local max nor local min
D. Unknown

______3. For what values of x is $f(x)=x+\frac{1}{x}$ decreasing?

A. $-1<x<1$
B. $x<-1$
C. $0<x<1$
D. all real numbers

______4. If $f(x)=-2x^3-6x^2+8$, then which of the following statements is **true?**

I. $f'(x)>0\ for\ x\in(-2,0)$
II. $f'(-2)=f'(0)=0$
III. $f(x)$ has one point of inflection

A. I only B. I and II only C. III only D. II and III only E. I, II & III

______5. A critical number is:

A. A number c in the domain of a function f such that $f'(c)>0$.
B. A number c in the domain of a function f such that $f'(c)<0$.
C. A number c in the domain of a function f such that $f'(c)=0$.
D. A number c in the domain of a function f such that $f'(c)=0$ or does not exist.
E. None of the above

6. True (T) or False (F).

(a) If local max point is (2,-9), then $f'(2)=0$

(b) A function has an absolute maximum at a if $f(a)\geq f(x)$ for all x in the domain.

(c) A local maximum point can also be an absolute maximum point but not vice-versa.

7. Find the absolute extreme values of the function $g(x)=x^{\frac{2}{3}}(5+x)$, if $-5 \le x \le 1$.

8. For Given the function $y = f(x)$ as shown, state

a) the interval(s) where

I. $f'(x)<0 \ \& \ f(x)>0$ ______________

II. $f'(x)>0 \ \& \ f(x)<0$ ______________

b) the local extrema values (specify the "type")

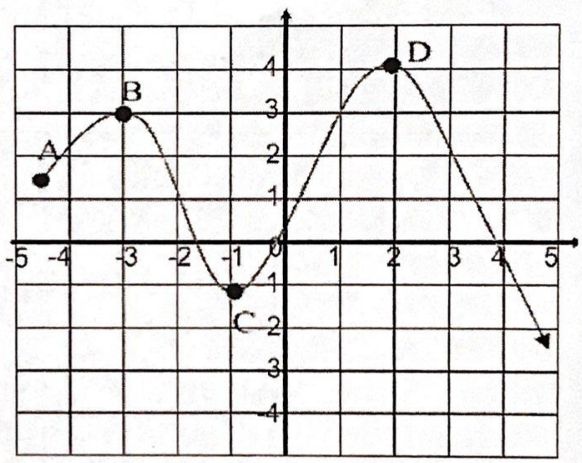
____

____

c) the points where $f'(x)=0$

d) absolute maximum: ______________

e) absolute minimum : ______________



9. Find the intervals on which the function $f(x)=(x^2-9)^{\frac{2}{3}}$ is increasing and the intervals on which it is decreasing. Determine the x-coordinates of all local extrema.

MHF4U7 Unit 4: Review

Curve Sketching and Optimization

Part A: Multiple Choice-Write the letter of your choice in the space at left.

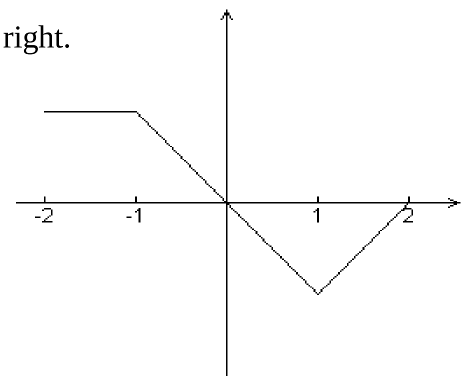
____ 1) Let *f* be the function with derivative given by $f'(x)=x^2-\frac{2}{x}$. On which of the following intervals is *f* decreasing?

A) $(-\infty,-1]$ B) $(-\infty,0)$ C) $[-1,0)$ D) $(0,\sqrt[3]{2})$ E) $(\sqrt[3]{2},\infty)$

____ 2) If the line tangent to the graph of the function *f* at the point $(1,7)$ passes through the point $(-2,-2)$, then $f'(1)$ is

A) –5 B) 1 C) 3 D) 7 E) undefined

____ 3) The graph of f', the derivative of the function *f*, is shown at right. Which of the following statements is true about *f* ?



Graph of f'

A) *f* is decreasing for $-1<x<1$.

B) *f* is increasing for $-2<x<0$.

C) *f* is increasing for $1<x<2$.

D) *f* has a local minimum at $x = 0$.

E) *f* is not differentiable at $x = -1$ and $x = 1$.

____ 4) Determine constants *a* and *b* such that the function $f(x)=x^3+ax^2+bx+c$ has a relative minimum at $x=4$ and a point of inflection at $x=1$.

A) $a = 1$, $b = 3$ B) $a = -3$, $b = 3$ C) $a = -3$, $b = -24$
D) $a = 3$, $b = 0$ E) $a = -6$, $b = 2$

____ 5) Which of the following are true for $h(x)=x^4-4x$?

I. *h* has a point of inflection at $x = 0$
II. *h* has an absolute minimum of -3
III. The second derivative test for local extrema is inconclusive

A) I and II B) I and III C) II and III D) I only E) II only

Part B: Full Solution

1. Find the exact value of a such that the function $f(x)=\sqrt{x-2}-\dfrac{a}{x}$ has a point of inflection at x=3.

2. Sketch $f(x)=x^{\frac{2}{3}}(8-x)^{\frac{1}{3}}$. Find the coordinates of all relative extrema and inflection Points.

$$f'(x)=\frac{16-3x}{3x^{\frac{1}{3}}(8-x)^{\frac{2}{3}}} \qquad f''(x)=\frac{-128}{9x^{\frac{4}{3}}(8-x)^{\frac{5}{3}}}$$

3. Perform a sketch for the following functions. Clearly indicate the results of each step.

a) $f(x)=\dfrac{x^3-4x}{3x^2+9x}$, $f'(x)=\dfrac{x^2+6x+4}{3(x+3)^2}$, $f''(x)=\dfrac{10}{3(x+3)^3}$

b) $f(x)=\dfrac{3x^2}{(x\text{-}3)^2}$, $f'(x)=\dfrac{\text{-}18x}{(x\text{-}3)^3}$, $f''(x)=\dfrac{18(2x+3)}{(x\text{-}3)^4}$

c) $f(x)=4-(x-3)^{\frac{2}{3}}$

4. Consider the function $f(x)=-x^3+6x^2+15x+2$ on the closed interval [-2, 2].

a) Find, and classify, all relative extrema.
b) Determine the absolute maximum of the function on the interval.

5. Let $f(x)=x\left(4+x^2-\dfrac{x^4}{5}\right)$ .Find the interval(s) on which f is increasing.

6. Use the **SECOND DERIVATIVE TEST** to find the local extrema of the function $f(x)=3x^4+8x^3-6x^2-24x+3$.

7. The function $f(x)=2kx^3+3x^2+px-3$ has a local minimum at x = -1 and a point of inflection at x =1 .Determine the values of k and p.

8. A function is defined by $f(x)=ax^3+bx+c$.

a) Find the values of a, b, and c if $f(x)$ has a y-intercept at (0, 2) and a local maximum at (2, 6).
b) Explain how you know there must be local minimum.

9. The slope of tangent to $g(x)=ax^3+bx^2+cx$ at its point of inflection (1, 5) is 4. What are the values of a, b, and c.

10. Sketch the graph of a rational function that satisfies all of the following conditions:

$$f''(x)<0 \text{ when } x<-4 \qquad f''(x)>0 \text{ when } x>-4 \qquad f'(x)<0 \text{ for all } x$$

$$\lim_{x\to -4^+} f(x)=+\infty \qquad \lim_{x\to -4^-} f(x)=-\infty \qquad \lim_{x\to \pm\infty} f(x)=-2 \qquad f(2)=0$$

11. a) Let f be a function that is even and continuous on the closed interval [-3,3]. The function f and its derivatives have the following properties:

$f(x) > 0$ when $0 < x < 1$ $f(x) < 0$ when $1 < x < 2$ and $2 < x < 3$

$f(0) = 1$, $f(1) = 0$, $f(2) = -1$, $f(3) = 0$

$f'(x) > 0$ when $2 < x < 3$, $f'(x) < 0$ when $0 < x < 1$ and when $1 < x < 2$

$f'(0) = DNE$, $f'(2) = DNE$, $f'(1) = 0$

$f''(x) > 0$ when $0 < x < 1$, $f''(x) < 0$ when $1 < x < 2$ and when $2 < x < 3$

$f''(0) = DNE$, $f''(2) = DNE$ and $f''(1) = 0$

b) Find all values of x at which f has a relative extremum. Justify your answer.

c) Find the coordinates of any inflection points on the graph of f.

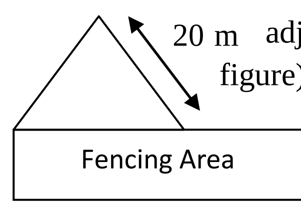
d) Identify the coordinates of any cusp or vertical tangent on the graph of f.

12. Find constants **a**,**b**, and **c** so that the slope of **normal** to the function $f(x) = ax^3 + bx^2 + c$ at its point of inflection (1,5) is $\frac{-1}{6}$.

13. Let $f(x) = \frac{x^2 - ax + 2}{x^2 - bx + 3}$. Determine the values of **a** and **b** so that f(x) has **only one** vertical asymptote at x = -3.

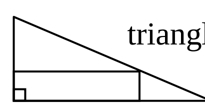
14. Two positive numbers have a product of 9. How should they be chosen so that the sum of their squares will be a minimum? What is that minimum?

15. Lucas has 60 m of fencing that he plans to use to enclose a rectangular area adjacent to an equilateral triangular shape garden which is 20 meter long (see the figure). Find the dimensions that will maximize the area.



16. What are the dimensions of the rectangle of largest area that has a diagonal of length 60 m?

17. What are the dimensions of the largest rectangle that can be inscribed in a right triangle with base 60 m and height 40 m?



18. Find the dimensions of the isosceles triangle of greatest area having a perimeter of 60 m.

19. A rectangular box-shaped garbage can with a square base and an open top is to be constructed using exactly 2700 cm^2 of material. Find the dimensions of the box that will provide the greatest possible volume.

20. The owner of a condominium complex has 45 units all of which will be occupied if the rent charged is \$600 per month. The owner estimates that for every \$20 increase in rent, one of the units will become vacant. The owner sets aside \$60 per month from each of the occupied units to establish a repair fund. What rent should be charged per month in order to maximize the owner's profit if there are no other expenses? What is the owner's maximum monthly profit? How many units are occupied?

21. If it costs \$1000 to manufacture 200 gizmos, then the average cost to manufacture each gizmo is $\$1000/200$ or \$5. Assume that the cost to manufacture *x* gizmos is given by the function $f(x) = 700 + 0.3x + 0.006x^2$. Find the number of gizmos that should be manufactured to minimize the average cost per gizmo.