Unit 5
Trigonometric, Exponential and Logarithmic Functions Coursepack

Trigonometry Warm Up

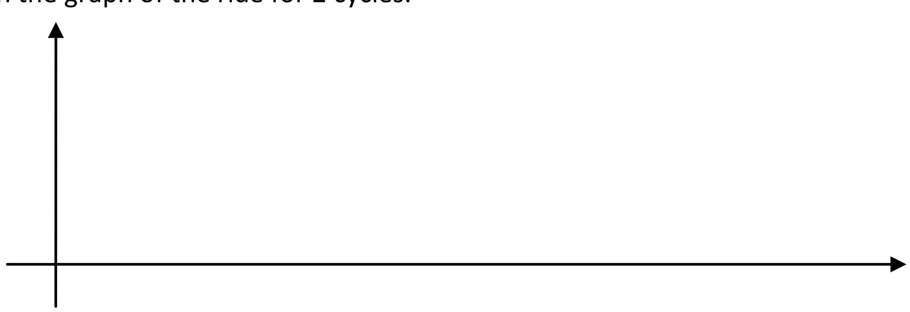
1. Determine the exact value of $\sin\frac{5\pi}{4}$.

2. Solve: $\cos\theta = \frac{1}{2}, 0 \leq \theta \leq 2\pi$

3. Solve: $2\sin^2\theta - \sin\theta = 0, 0 \leq \theta \leq 3\pi$

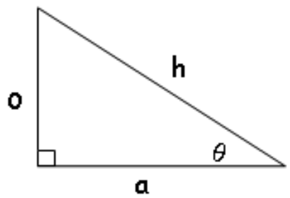
4. Prove: $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

5. A Ferris Wheel has a radius of 10 m and completes one full revolution in 36 s. The riders board the ride from a platform 1 m above the ground at the bottom of the wheel.
   a) Determine the equation of the sine function that models the position of a rider above the ground, $h(t)$, in metres, at time t, in seconds.
   b) When will the rider be at maximum height?
   c) What is the maximum height?
   d) Sketch the graph of the ride for 2 cycles.



5.1: Part I-Trigonometry –Double Angle Formulas

Primary Trigonometry Ratio (valid for right angle triangles only)



SINE	COSINE	TANGENT
$\sin(\theta)=\frac{o}{h}$	$\cos(\theta)=\frac{a}{h}$	$\tan(\theta)=\frac{o}{a}$

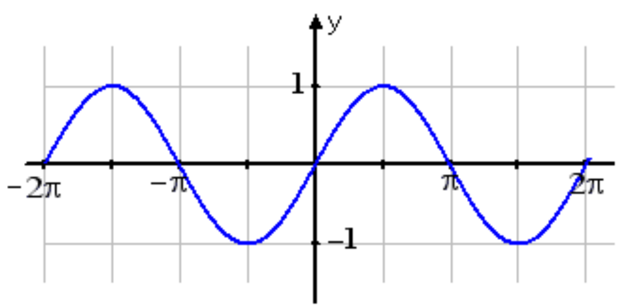
Reciprocal Trigonometric Ratios

The reciprocals of the primary trigonometric ratios sine, cosine and tangent are cosecant, secant and cotangent, respectively.

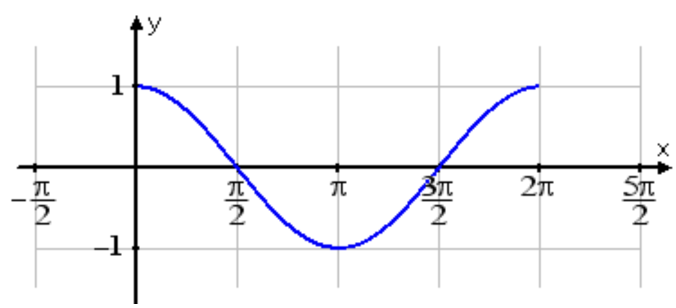
COSECANT	SECANT	COTANGENT
$\csc(\theta)=\frac{1}{\sin(\theta)}$	$\sec(\theta)=\frac{1}{\cos(\theta)}$	$\cot(\theta)=\frac{1}{\tan(\theta)}$

Graphs

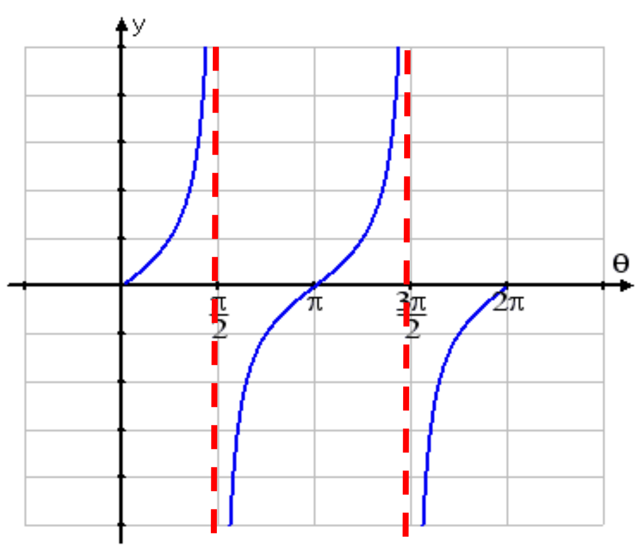
$y=\sin(\theta),\ Period=2\pi$



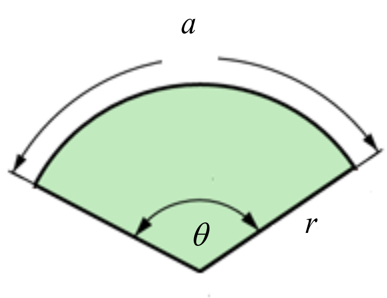
$y=\cos(\theta),\ Period=2\pi$



$y=\tan(\theta),\ Period=\pi$

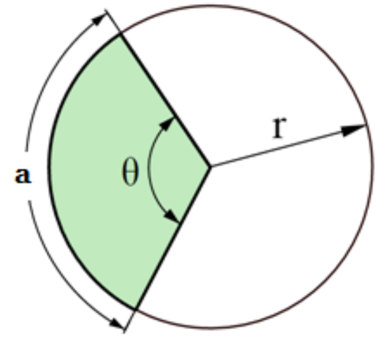


Area of Sector

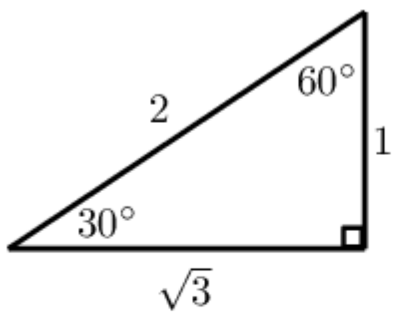


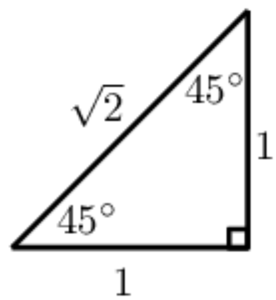
$$A = \frac{1}{2} r^2 \theta$$

$$\theta = \frac{a}{r}$$

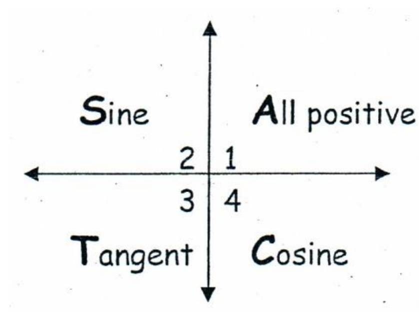


Special Angles





CAST Rule



Reciprocal Identities

$$\csc A = \frac{1}{\sin A}$$

$$\sec(A) = \frac{1}{\cos(A)}$$

$$\cot(A) = \frac{1}{\tan(A)}$$

$$\tan(A) = \frac{1}{\cot(A)}$$

$$\cot(A) = \frac{1}{\tan(A)}$$

Quotient Identities

$$\tan(A) = \frac{\sin(A)}{\cos(A)}$$

$$\cot(A) = \frac{\cos(A)}{\sin(A)}$$

Pythagorean Identity

$$\sin^2(A) + \cos^2(A) = 1$$

$$\tan^2(A) + 1 = \sec^2(A)$$

$$\cot^2(A) + 1 = \csc^2(A)$$

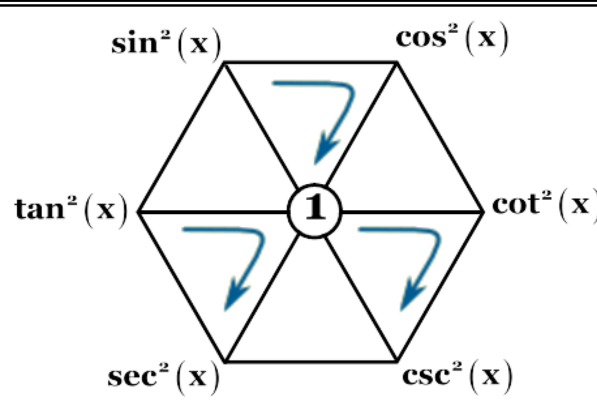
Reflection Identities

$$\sin(-A) = -\sin(A)$$

$$\cos(-A) = \cos(A)$$

$$\tan(-A) = -\tan(A)$$

The Hexagon Trig Trick



COMPOUND ANGLE IDENTITIES

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

DOUBLE ANGLE IDENTITIES

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\cos(2A) = 2\cos^2(A) - 1$$

$$\cos(2A) = 1 - 2\sin^2(A)$$

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

Example 1: Express the following as a single trigonometric ratio.

a) $2\sin(5A)\cos(5A)$

b) $1-2\sin^2(3x)$

c) $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$

d) $\sin(2x)\cos(2x)$

e) $2\cos^2(3\theta-2)-1$

f) $\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)$

Example 2: Express the following as a single trigonometric ratio and then evaluate.

a) $\cos^2\left(\frac{\pi}{12}\right)-\sin^2\left(\frac{\pi}{12}\right)$

b) $\dfrac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$

Example 3: If $\cos(x)=\frac{3}{5}$, find the exact value of $\cos(4x)$.

Example 4: If $\cos(x) = -\frac{3}{5}, \frac{\pi}{2} < x < \pi$, find the exact value of $\sin(2x)$.

Example 5: Determine the exact value of $\cos\left(\frac{\pi}{8}\right)$.

Practice

1. **Multiple Choice:** Select the best answer for each of the following

_____ i. Given $\cos(\theta)=\frac{2}{3}$ in the first quadrant, the value of $\sin\left(\frac{\theta}{2}\right)$ is:

a. $\frac{\sqrt{3}}{2}$ b. $\frac{\sqrt{6}}{2}$ c. $\frac{\sqrt{6}}{6}$ d. $\frac{\sqrt{3}}{6}$

_____ ii. $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ is equivalent to :

a. $\sin\left(\frac{\pi}{8}\right)$ b. $\cos\left(\frac{\pi}{8}\right)$ c. $\sin\left(\frac{\pi}{4}\right)$ d. $\cos\left(\frac{\pi}{4}\right)$

_____ iii. The exact value of $1-2\sin^2\left(\frac{\pi}{8}\right)$ is:

a. $-\sqrt{2}$ b. $\frac{\sqrt{2}}{2}$ c. $-\frac{\sqrt{2}}{2}$ d. $\sqrt{2}$

2. Express as a single sine or cosine function.

a) $10\sin(x)\cos(x)$

b) $1-2\sin^2\left(\frac{2\theta}{3}\right)$

c) $5\sin(2x)\cos(2x)$

d) $2\cos^2(5\theta)-1$

3. Simplify each expression.

a) $\frac{\sin(2a)}{\cos(a)}=$

b) $2\tan(a)\cos^2(a)=$

c) $2\sin^2(a)+\cos(2a)=$

4. Expand using a double angle formula.

a) $3\sin(4x)=$

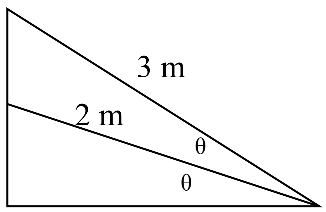
b) $6\cos(6x)=$

c) $1-\cos(8x)=$

d) $\tan(4x)=$

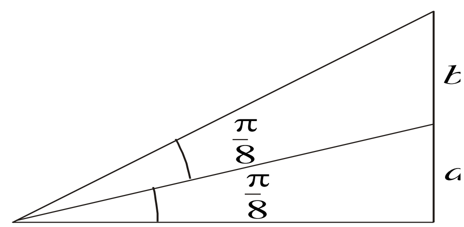
e) $\cos(2x)-\frac{\sin(2x)}{\sin(x)}=$

5. Two ropes (2m and 3m long) used to stabilize a pole for a volleyball net are anchored to the ground. The angle between the two ropes is equal to the angle between the ground and the lower rope. Determine the distance from the base of the pole to the point at which the ropes are anchored to the ground.



6. Express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of $\tan(\theta)$.

7. Find the exact value of b.



Warm Up

1. The expression $4\cos^2\left(\frac{3}{2}\theta\right)-2$ expressed as a single trig function is:

   A. $\cos\left(\frac{3}{2}\theta\right)$ B. $2\cos(2\theta)$ C. $2\cos(3\theta)$ D. $2\cos\left(\frac{3}{2}\theta\right)$

2. The exact value of the $\dfrac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$ is:

   A. $\dfrac{\sqrt{3}}{3}$ B. $\sqrt{3}$ C. 1 D. $\dfrac{\sqrt{2}}{2}$

3. If $4\tan(2\theta)-3=0, 0<\theta<\frac{\pi}{4}$, and $13\sin(\alpha)+5=0, 0<\alpha<\frac{3\pi}{2}$, determine the **exact** value of $\tan(\theta)+\sin(2\alpha)$.

4. Determine the **exact** value of $\cos^2\left(\frac{7\pi}{8}\right)$.

5.1 Part II- Proving Trigonometric Identities

Fill in the blanks with the words in the box.

Counter-example	**trig-identity** **identity**	**equal**

- A statement of equality between two expressions that is true for all values of the variables for which the expressions are defined is called a(n) ________________.
- An identity involving trigonometric expressions is called a __________________.
- Our goal is to prove that one side of an expression is __________ to the other side of the expression.
- A ________________ can be used to show that an equation is not an identity

Strategies for Proving Trig Identities:

- Write everything in terms of sine and cosine
- Be aware of equivalent forms of the fundamental identities, ie: $\sin^2(\theta)+\cos^2(\theta)=1$ has an alternative form: $\sin^2(\theta)=1-\cos^2(\theta)$
- Try to rewrite the more complicated side of the equation so that it is identical to the simpler side.
- Usually any factoring or indicated algebraic operations should be performed, ie:

  $1+\sin(2x)=(\sin(x)+\cos(x))^2$ or

  $$\sin^3(x)+\cos^3(x)=(\sin(x)+\cos(x))\left(\sin^2(x)-\sin(x)\cos(x)+\cos^2(x)\right)$$
  $$=(\sin(x)+\cos(x))\left(1-\frac{1}{2}\sin(2x)\right)$$
- If an expression contains $1\pm\sin x$, $\sec x\pm\tan x$ or $\csc x\pm\cot x$ multiplying both numerator and denominator by $1\mp\sin x$, $\sec x\mp\tan x$ or $\csc x\mp\cot x$ would give $\cos^2 x$,1 or -1.

1. Prove the following identities:

a) $\dfrac{1+\sec(x)}{\tan(x)+\sin(x)}=\csc(x)$

b) $\sin(2A)=\dfrac{2\tan(A)}{1+\tan^2(A)}$

c) $\cos^4(x) - \sin^4(x) = \cos(2x)$

d) $\cos^4(A) + \sin^4(A) = 1 - \frac{1}{2}\sin^2(2A)$

e) $\frac{1 - \sin(2x)}{\cos(2x)} = \frac{1 - \tan(x)}{1 + \tan(x)}$

f) $2\cos^3(x) + \sin(2x)\sin(x) = 2\cos(x)$

PRACTICE

1. Prove each identity.

a) $\frac{\sec(\theta)-1}{1-\cos(\theta)}=\sec(\theta)$

b) $\tan(x)+\cot(x)=\frac{2}{\sin(2x)}$

$\frac{\sin(t)-\cos(t)}{\cos(t)}+\frac{\sin(t)+\cos(t)}{\sin(t)}=\sec(t)\csc(t)$

c) $\frac{1+\tan^2(x)}{1+\cot^2(x)}=\frac{1-\cos^2(x)}{\cos^2(x)}$

d) $\frac{1}{1+\sec(\theta)}+\frac{1}{1-\sec(\theta)}=-2\cot^2(\theta)$

e) $\frac{1+\sec(x)}{\tan(x)+\sin(x)}=\csc(x)$

f) $\cos(a+b)\cos(a-b)=\cos^2(a)-\sin^2(b)$

g) $\frac{\cos(x)-\sin(y)}{\cos(y)-\sin(x)}=\frac{\cos(y)+\sin(x)}{\cos(x)+\sin(y)}$

h) $\frac{1+\tan(A)}{\sin(A)}-\sec(A)=\csc(A)$

i)

j) $\frac{[1+\cos(2x)][1+\cos(x)]}{[\sin(2x)][\cos(x)]}=\cot\left(\frac{x}{2}\right)$

k) $\frac{4-\sin^2(2x)}{4\cos^4(x)}=\tan^4(x)+\tan^2(x)+1$

l) $1-\sin(x)\cos(x)=\frac{\sin^2(x)}{1+\cot(x)}+\frac{\cos^2(x)}{1+\tan(x)}$

m) $\frac{\sin(x-y)}{\sin(x)\sin(y)}=\cot(y)-\cot(x)$

n) $\frac{\sec(x)+\tan(x)}{\sin(x)}=\frac{\csc(x)}{\sec(x)-\tan(x)}$

2*. If $2\cos^2(x)+4\sin(x)\cos(x)$ is expressed in the form $A\sin(2x)+B\cos(2x)+C$ where $A,B,C\epsilon R$, determine the values of A, B, and C.

3*. Write $2\sin(2x)+\sqrt{12}\cos(2x)$ in the form $y=A\cos(2x-\theta)$ by finding $A>0$ and $\theta\in[0,2\pi]$.

5.1 Part III Solving Trigonometric Equations

What is a trig equation?

- A trig equation is an equation that contains one or more trigonometric functions.

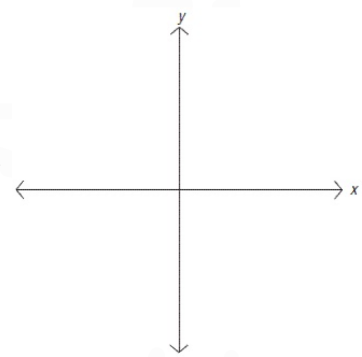
How is it similar to a trig identity?

- A trig equation *can* be but *does not have to* be a trig identity
  - A trig identity is an equation that is true for ***all values*** of the variable for which expressions of both sides of the equation are defined
- A trig equation that is not an identity is only true for ***certain values*** of the variable.

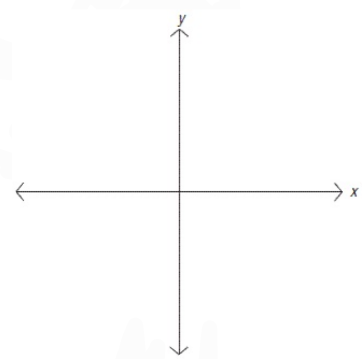
What does it mean to **SOLVE** a trig equation?

- Much like solving a linear equation, we are looking for all of the values of the variable that makes the equation true.

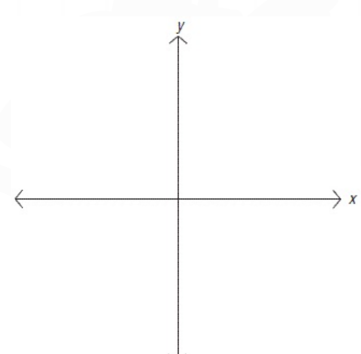
Ex.1) Find the exact solutions for $2\cos(x)+\sqrt{3}=0$ for $x\in[0,2\pi]$.



Ex.2) Find the exact solutions for $2\sin(2x)=\sqrt{3}$ for $x\in[0,2\pi]$.



Ex.3) Find the exact solutions for $3\tan^2(\theta)=1$ for $\theta\in[0,2\pi]$.



Ex.4) Find the exact solutions for $\tan^2(\theta)+3\sec(\theta)+3=0$, $0\le\theta\le 2\pi$

Ex.5) Find the solutions for $3\,cos^2(\theta)-\sin(\theta)-1=0$, $0\le\theta\le 2\pi$

Ex.6) Solve the equation $\sin(4x)-\cos(2x)=0$, $x\in[0,2\pi]$

Ex.7) Solve $\sin(2x) - \cos(2x) = 0, \quad 0 \le x \le 2\pi$

Ex.8) Solve $4\cos(2x) - \sin(x)\csc^3(x) + 2 = 0, 0 \le x \le 2\pi$

Practice

1. Solve for x on the interval $[0, 2\pi]$.

a) $6\sin^2(x) - \sin(x) - 1 = 0$ b) $\cot(x)\cos^2(x) = \cot(x)$ c) $4\tan(x) - \sec^2(x) = 0$

d) $2\sin^2(x) - \sin(x) - 1 = 0$ e) $4\sin^3(x) + 2\sin^2(x) - 2\sin(x) - 1 = 0$ f) $\sin(x) + \sqrt{2} = -\sin(x)$

g) $2\cos(3x - 1) = 0$ h) $\sin(2x)\cos(x) - \cos(2x)\sin(x) = 0$ i) $\sec^2(x) - 2\tan(x) = 4$

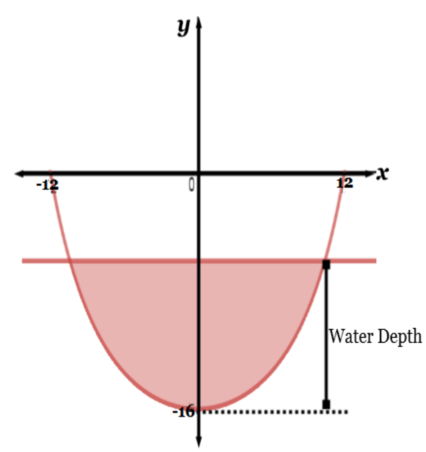
j) $3\tan\left(\frac{x}{2}\right) + 3 = 0$ k) $-6\sin(2x)\cos(x) + 8\cos(2x) + 3\sin(x) + 4 = 0$

l) $\cot(x)\cos^2(x) = 2\cot(x)$ m) $\frac{1 + \sin(x)}{\cos(x)} + \frac{\cos(x)}{1 + \sin(x)} = 4$ n) $2\sin^2(x) + 3\cos(x) - 3 = 0$

o) $2\sin(x)\tan(x) - \tan(x) - 2\sin(x) + 1 = 0$ p) $\cos(x)\tan(x) - 1 + \tan(x) - \cos(x) = 0$

2. A weight hanging from a spring is set in motion moving up and down. Its distance, d, (in cm) above or below its "rest" position is described by $d(t) = 5(\sin(6t) - 4\cos(6t))$. At what times during the first 2 seconds is the weight at the rest position (d=0).

3. The diagram below shows the boundary of the cross section of a water channel. The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm. Find the width of the water surface when the water depth in the channel is 10 cm. [Round your answer to 2dp] .



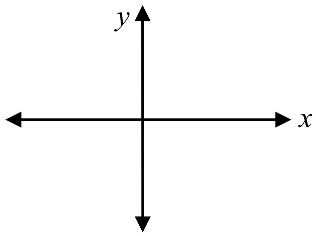
4. Solve $\frac{\sin^2(x) - 3\sin(x)}{1 - 2\cos(x)} \geq 0$ over the interval $0 < x < 2\pi$.

5. Solve $\sin x = \frac{x}{2}$, $0 \leq x \leq 2.5$, $x \in \mathbb{R}$ (Use GDC).

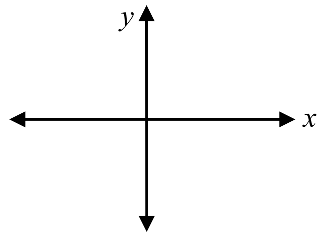
Derivatives of the Sine, Cosine and Tangent Functions

Using the GDC, determine the following:

a) $\lim_{x \to 0} \frac{\sin x}{x} =$



b) $\lim_{x \to 0} \frac{\cos x - 1}{x} =$



Using the above limits, the identity: $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, and first principles, determine the derivative of $y = \sin(x)$.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Therefore, the derivative of $y = \sin(x)$ is __________________ (or__________________________).

In general, if $y = [\sin(g(x))]^n$, then $y' =$____.

Using the previous limits, the identity: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, and first principles, determine the derivative of $y = \cos(x)$.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

Therefore, the derivative of $y = \cos(x)$ is _________________ (or__________________________).

In general, if $y = [\cos(g(x))]^{n}$, then $y' =$____.

Determine the derivative of $y = \tan(x)$.

Therefore, the derivative of $y = \tan(x)$ is ________________ (or________________________).

In general, if $y = [\tan(g(x))]^{n}$, then $y' =$____.

Determine the derivative of the following:

a) $y = \csc(x)$

b) $y = \sec(x)$

c) $y = \cot(x)$

Derivatives of Secondary Trigonometric Ratios

In general,

$\frac{d}{dx}\csc(x)=$

$\frac{d}{dx}\sec(x)=$

$\frac{d}{dx}\cot(x)=$

Example 1: Differentiate the following functions:

a. $y = sin^2(x)$

c. $y = \sec(x)$

b. $y = \sin(x^2)$

d. $y = (\tan(3x))\sqrt{\cos(x)}$

Example 2: Find the equation of the line that is tangent to the curve $y = \dfrac{\cos(x)}{x}$ at the point on the curve where $x = \pi$

Example 3: Find all extrema for $y = 2x - \tan(x)$, $x \in [0, \pi]$

Example 4: Find the derivative: $y = \tan(\sin(\cos(x^2)))$

Example 5: Find all local extrema for $y = 2cos^2(x) + 1$, $x \in [0, 2\pi]$

Practice Questions 5.2: **Derivatives of Trigonometric Functions**

Multiple Choice. *Identify the choice that best completes the statement or answers the question.*

1. If $f(x) = 2cos(3x)$, find $f'(\frac{\pi}{3})$.

a. 0 b. 3 c. 6 d. –6

2. If $f(x) = sinx + cosx + x$, find $f'(\pi)$.

a. 2 b. π c. 0 d. $1 + \pi$

3. If $y = tan2x$, find y'' at $x = \frac{\pi}{2}$.

a. $\frac{\sqrt{3}}{8}$ b. $8\sqrt{3}$ c. –8 d. 0

4. Find the slope of the tangent to $y = cos^2x$ at the point $(\frac{\pi}{3}, \frac{1}{4})$.

a. $-\frac{\sqrt{3}}{2}$ b. 1 c. 2 d. $\frac{\sqrt{3}}{2}$

5. Find the slope of the tangent to $y = x^2\cos(4x^2 + 7)$ at the point where x=1.

a. 6.011 b. 3.023 c. –4.203 d. 8.009

6. Differentiate the following:

a. $f(x) = cos(3x^2)$	b. $y = sin\sqrt{x^2 - 1}$
c. $y = (2x^2 - 4x)cos^3(2x)$	d. $y = (1 - 3cos^2x)^4$
e. $f(x) = (sin3x + cos3x)^4$	f. $y = \frac{\tan(3x)}{\sin(2x)}$

7. If $y = A(coskt) + B(sinkt)$, where A, B, and k are constants, show that: $y'' + k^2 y = 0$.

8. Prove that $y = secx + tanx$ is always increasing on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

5-4 Warm Up

1. Solve for x: $\sqrt{3}\cos(x) = sin\ (2x)$, $-2\pi \leq x \leq 2\pi$

2. Solve: $\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) = \sqrt{2}$, $0 \leq x \leq 2\pi$

3. Prove the following identities:

   a) $\frac{1}{\cos(x)\sin(x)} + \frac{1}{\cos(x)-\sin(x)} = \frac{2\cos(x)}{\cos(2x)}$

   b) $2\cos^3(x) + \sin(2x)\sin(x) = 2\cos(x)$

4. Differentiate with respect to x:

   a) $y = \cos^3(x)$

   b) $y = \sin(x)$

   c) $y = x^3 + \sin^2(x)$

   d) $y = \sin^4(x^3 - 1)$

5. Find the equation of the tangent line to the curve $f(x) = 6\tan(x) - \tan(2x)$ at $x = 0$.

Warm-up: Derivatives of sin x, cos x, and tan x

1. Find $\frac{dy}{dx}$ for each of these functions.

a) $y = 2\sqrt{\cos 4x}$

b) $y = \left(1 + \sin\sqrt{x}\right)^3$

c) $y = \frac{3}{5}\tan(x^5)$

d) $y = 2\sin^2 x \cos 3x$

e) $y = \frac{5}{4 - 3\tan x}$

f) $x + y\sin x^2 = 1 - \cos^3 y$

2. Find the equation of the normal to the curve $y = x + \tan x$ at the point where $x = \frac{\pi}{4}$.

Differentiatation Rules for Trigonometric Functions

Recall:

- The Product Rule: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
- The Chain Rule: $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\frac{d}{dx}[g(x)]$
- The Quotient Rule: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

Ex. 1. Differentiate with respect to x.

a) $y = \sin(2x^3)$

b) $f(x) = \frac{-1}{4}\csc(-8x)$

c) $y = \cos^3(x^2 - 1)$

d) $y = \cot^3(1 + x^2)$

Ex. 2. Find the equation of the tangent line to $y = \dfrac{\sin x}{\cos x}$ at $x = \dfrac{\pi}{3}$.

Application of derivatives of trigonometric functions

A point on a curve at which the gradient is zero, where $\frac{dy}{dx} = 0$, is called a **stationary point.**

Three types: Minimum point, Maximum point, Point of inflexion

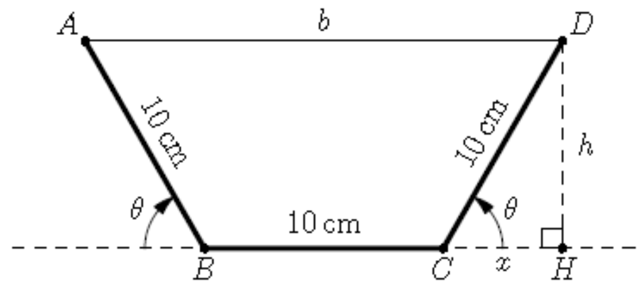
1. Find and classify the stationary points on the curve $y = x + \sin x$ in the range $0 \le x \le 2\pi$. Sketch the curve using the GDC.

2. Given that $y = \sin x(1 - \cos x)$, show that $\frac{dy}{dx} = (1 + 2\cos x)(1 - \cos x)$

Hence, find the coordinates of the points on the curve $y = \sin x(1 - \cos x)$, in the range $0 \le x \le \pi$, where the gradient is zero.

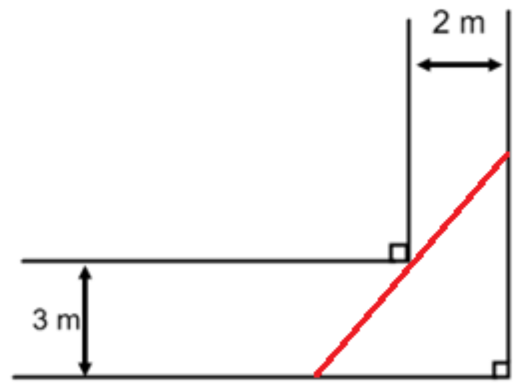
Optimization Problems with Trigonometric Functions

Ex 1: A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side through an angle, θ. How should θ be chosen so that the gutter will carry the maximum amount of water?

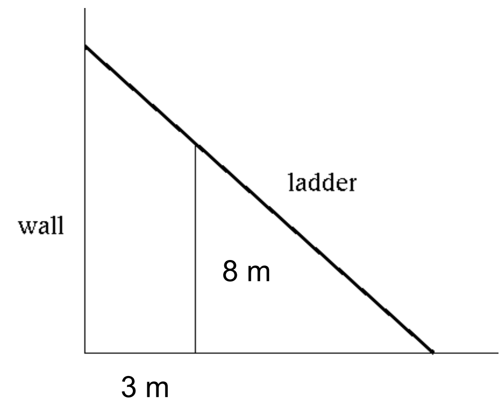


Ex 2: A ladder is to be carried down a hallway *2* m wide. Unfortunately at the end of the hallway there is a right-angled turn into a hallway *3* m wide. What is the length of the longest ladder that can be carried **horizontally** around the corner?

The goal is shifted from finding the longest ladder that will go around the corner to finding the shortest ladder that will get stuck.



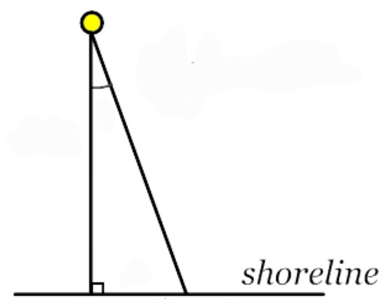
Ex 3. A fence 8 m tall runs parallel to a house at a distance of 3 m from the wall of a house. Determine the angle, θ, the shortest ladder will make from the foot of the ladder to the ground, if the ladder is to reach from the ground over the fence and to the wall of the house



Related Rates Problems with Trigonometric Functions

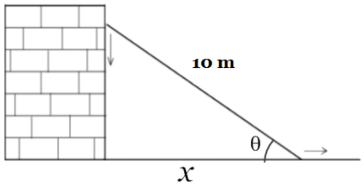
General Method for solving related rates problems: 1. Draw a picture and give names to all quantities involved. 2. Which rate do we want? Which rate do we know? 3. Find relationships between the quantities that are true for all times in the problem. The goal is to have a complete chain between the quantities whose rate we know and the quantity whose rate we want. 4. Take a derivative with respect to time of both sides of every relationship. 5. Plug in the specific values for the moment in time we care about. Do not do this step early. 6. Solve for the rate we want.

1. A lighthouse is located on a small island 3 km away from the nearest point *P* on a straight shoreline. If the lighthouse beacon rotates at a constant rate of 4 revolutions per minute, how fast is the beam of light moving along the shoreline when it is 1 km from the nearest point *P*?

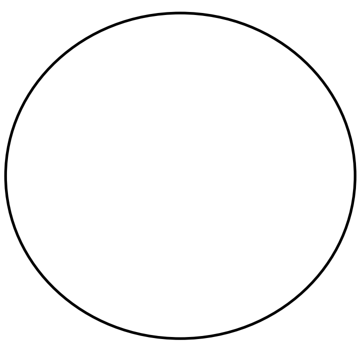


2. If the height of an isosceles triangle with base 2m changes at a rate 3 m/s, how quickly is the angle opposite the base changing when $h = \sqrt{3}$ m?

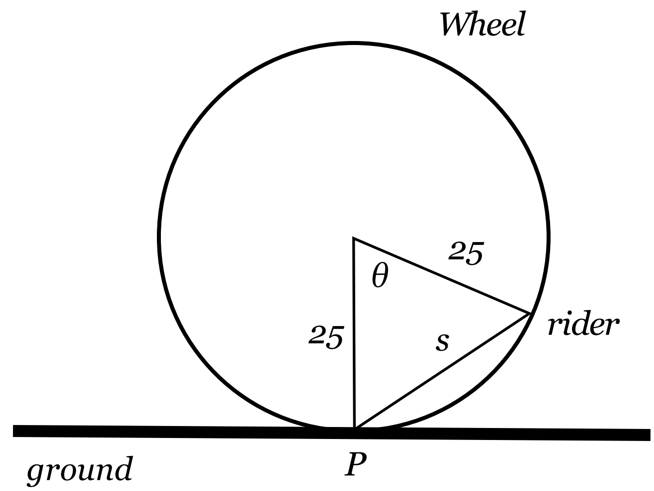
3. A 10-m ladder leans against a house on flat ground. The house is to the left of the ladder. The base of the ladder starts to slide away from the house at 2 m/s. At what rate is the angle between the ladder and the ground changing when the base is 8 m from the house?



4. A runner sprints around a circular track of radius 100 meters at a constant speed of 7 meters per second. An observer is standing 200 meters from the track's center. How fast is the distance between the runner and the observer changing when the distance between them is 200 meters?



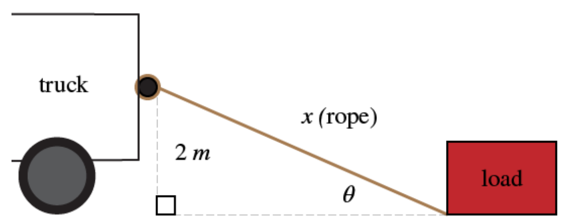
5. A Ferris wheel 50 meter in diameter makes 2 revolutions per minute. Assume that the wheel is tangent to the ground and let P be the point of tangency. At what rate is the distance between P and a rider changing, when she is 15 meter above the ground and going up?



Related Rates Practice

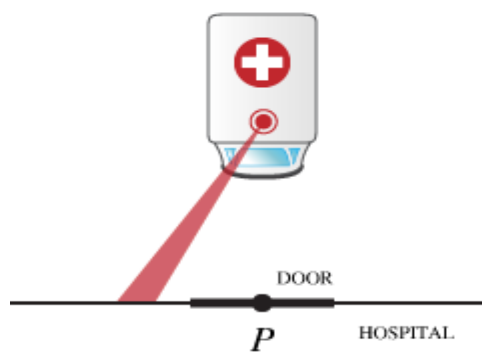
1. A rotating beacon is located 2 km out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 km along the shore from the point A? [Ans. 80π]

2. A winch on the back of a stationary tow truck is used to drag a large load along the ground. The winch is located 2 m above the ground and cranks the cable in at a rate of 0.75 m/s. At what rate is the angle between the cable and the ground changing when the angle is equal to $\frac{\pi}{4}$? [Ans. $\frac{3\sqrt{2}}{16}$ rad / sec]

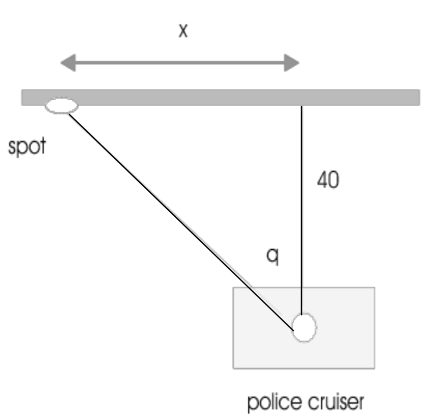


3. Two sides of a triangle have lengths 8 cm and 15 cm. The angle, θ, between them is increasing at a rate of 160 radians per second. It follows that the length of the third side, c, is also increasing with respect to time.
   (a) Use the cosine law to find an equation relating the angle θ to the length of the third side, c. [Ans. $c^2=289-240\cos(\theta)$]
   (b) Find the length of the third side, c, when $\theta=\frac{\pi}{2}$. [Ans. $c=17$]
   (c) At what rate is the length of the third side, c, changing when $\theta=\frac{\pi}{2}$? [Ans. $\frac{2}{17}$ cm / s]

4. The light on the top of an ambulance rotates and makes one revolution every 5 seconds. The light runs along a wall on the side of the hospital as it rotates. If the light is situated 10 metres from the wall, then how fast is the spot of light moving along the hospital wall when the spot is 3 metres from the centre of the door, located at point P?[Ans. $\frac{109\pi}{25}$ m / s]



5. A spotlight on the top of a police cruiser makes one revolution per second. The spotlight is 40 meter from a long straight wall. At what rate is the spot of light moving across the wall at the instant when the beam makes an angle of $\frac{\pi}{6}$ with the wall?[Ans. 320π m/s]

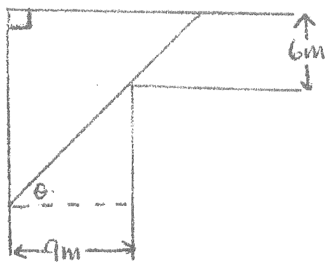


Related Rates & Optimization Involving Trigonometric Derivatives

Example1: A ladder 10m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2m/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ rad?

Example 2: A television camera is positioned 4000 m form the base of a rocket launching pad. If the camera is aimed at the rocket that is travelling at a speed of 600m/s, determine how fast the camera's angle is changing the moment the rocket has reached a height of 3000m.

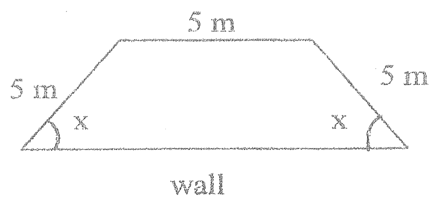
Example 3: A steel pipe is being carried down a hallway 9 m wide. At the end of the hall there is a right angled turn into a narrower hallway 6 m wide. What is the length of the longest pipe that can be carried horizontally around the corner?



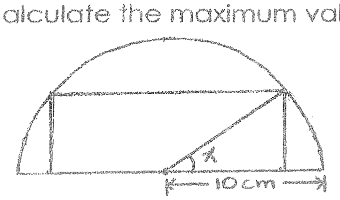
Application of derivatives of trigonometric functions

1. The height, h metres, of a Ferris wheel above the ground at time t seconds is given by $h = 20 + 20\sin\left(\frac{t+\pi}{4}\right)$. Find the rate at which its height is changing when t = 10 seconds.

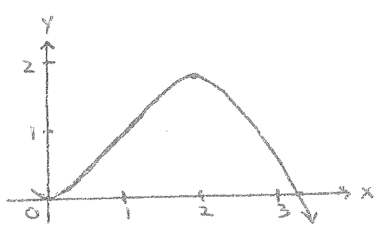
2. A farmer has three pieces of fencing, each of length 5 metres, and wishes to enclose a pen, as in the diagram. A long wall, which is already in place, will comprise the fourth side of the pen.
a) Show that the area of the pen is $[25\sin x(1+\cos x)]$ m².
b) Calculate the maximum value of this area.



3. A rectangle is drawn inside a semicircle of base radius 10 cm as in the diagram.
a) Show that the area of the rectangle is $200\sin x\cos x$ cm².
b) Calculate the maximum value of this area.



4. The diagram shows the graph of the function $g : x \to x\sin x$ for $-0.5 \le x \le 3.5$. The graph intersects the x-axis at the point A with coordinates (a, 0), and B is the maximum point on this part of the graph.
a) Find the value of a.
b)i) Find g'(x).
ii) Show that, at the maximum point B, x satisfies the equation $x + \tan x = 0$.
c) Show that the second derivative $g''(x) = -x\sin x + 2\cos x$.



Applications of Trigonometric Functions

1. A wall of height 8 m stands parallel to and 27 in from a tall building. A ladder with its foot on the ground is to pass over the top of the wall and lean on the side of the building. What angle will the shortest such ladder make with the ground?

2. A kite 40 m above the ground moves horizontally at the rate of 3 m/s. At what rate is the angle between the string and the horizontal decreasing when 80 m of string has been released?

3. A revolving beacon is situated 925 m from a straight shore. It turns at 2 rev/min.
   a) How fast does the beam sweep along the shore at its nearest point?
   b) How fast does it sweep along the shore at a point 1275 m from the nearest point?

4. Two sides of a triangle are six and eight meters in length. The angle between them decreases at the rate of 0.035 radian/s. Find the rate at which the area is decreasing when the angle between the sides of fixed length is $\frac{\pi}{6}$.

5. A ladder l0 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 m/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$?

6. The base of an isosceles triangle is 20 cm and the altitude is increasing at the rate of 1 cm/min. At what rate is the base angle increasing when the area is 100 cm^2?

7. A vehicle moves along a straight path with a speed of 4 m/s. A searchlight is located on the ground 20 m from the path and is kept focused on the vehicle. At what rate (in rad/s) is the searchlight rotating when the vehicle is 15 m from the point on the path closest to the searchlight?

Answers:
1. 0.588
2. 0.05 rad/s
3. a)11624 m/min; b) 33708 m/min
4. 0.727 m^2/s
5. $\frac{\sqrt{2}}{5}$ rad/s
6. 0.05 rad/min
7. 0.128 rad/s

Practise, Apply, Solve: Derivatives of trigonometric functions

1. Find the equation of the tangent to the curve $y = x + \sin x$ at the point where $x = \frac{\pi}{3}$.

2. Find the coordinates of the two points on the curve $y = \sin x(2\cos x + 1)$, in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, where the gradient is $-\frac{1}{2}$.

3. Given that $y = \frac{1}{\cos x} + \frac{1}{\sin x}$, show that $\frac{dy}{dx} = \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x}$

Hence find the coordinates of the point on the curve $y = \frac{1}{\cos x} + \frac{1}{\sin x}$, in the range $0 \leq x \leq \pi$, where the gradient is zero.

4. Find and classify the stationary values on each of the following curves in the range $0 \leq x \leq 2\pi$. Sketch each curve using your GDC.

a) $y = x + 2\cos x$

b) $y = \frac{\sin x}{2 - \sin x}$

5. The function $f(x) = a\sin x + b$ has a derivative $f'(x) = 3\cos x$. The point (0, 2) lies on the graph of the function. Find the values of a and b.

6. The point $P\left(\frac{1}{3}, 0\right)$ lies on the graph of the equation $y = \sin(3x - 1)$.

a) Find $\frac{dy}{dx}$.

b) Hence find the gradient of the tangent to the graph at P.

c) How many tangents parallel to this one are there between x = 0 and x = 2π?

7. A right-angled triangle has hypotenuse 15 cm and one angle x radians.

a) Find an expression for the perimeter of the triangle in terms of x.

b) Find an expression for the rate of change of the perimeter as x varies.

c) Hence find the value of x when this rate is greatest.

Answers:

1. $9x - 6y + 3\sqrt{3} - \pi = 0$
2. $\left(-\frac{\pi}{3}, -\sqrt{3}\right), \left(\frac{\pi}{3}, \sqrt{3}\right)$
3. $\left(\frac{\pi}{4}, 2\sqrt{2}\right)$

4. a) $\left(\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3}\right)$, max; $\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)$, min b) $\left(\frac{\pi}{2}, 1\right)$, max ; $\left(\frac{3\pi}{2}, -\frac{1}{3}\right)$, min

5. a = 3, b = 2

6. a) $3\cos(3x - 1)$ b) 3 c) 3 in total; at $x = \frac{1}{3}$ (given), $x = \frac{2\pi + 1}{3}$ and $x = \frac{4\pi + 1}{3}$

7. a) $15(1 + \cos x + \sin x)$ b) $15(\cos x - \sin x)$ c) $\frac{\pi}{4}$

Warm up

1. Find $\frac{dy}{dx}$ for each of the following. **Completely** simplify your answers.

a) $y = \cos(x^2+1)^3 + \sin^2(2\pi)$

b) $y = \cos^2(\sqrt{x}) - \sin^2(\sqrt{x})$

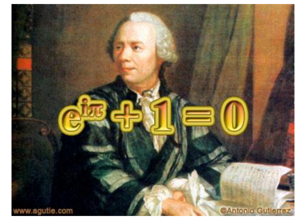
2. The position of a particle, in metres, is described by $s(t) = 10\cos\left(5t - \frac{\pi}{4}\right)$, for $0 \le t \le \pi$, where *t* is time, in seconds.

a) Determine the position of the particle when $t = 2$ seconds.

b) Determine when the velocity is zero.

The Number e and the Derivatives of $y = e^x$ and $y = lnx$

Introducing a Special Number, e



Leonhard Euler (1707-1783) was a remarkable Swiss mathematician and physicist. He made massive contributions to mathematics, especially calculus, as well as physics, optics, magnetism, astronomy, and shipbuilding. Euler popularized the use of the symbol π and developed new approximations for it. He was the first to use the symbol i to represent imaginary numbers. Euler also developed the irrational number e, which is known as Euler's number and is defined as a limit:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

Euler's Number e :

Let's examine some integer values of x to see the limiting value of this expression.

x	$\left(1+\frac{1}{x}\right)^x$
1	2
10	2.5937
100	
1000	
10000	
100000	
1000000	

We see that as x becomes larger, the value of the expression changes by a smaller and smaller amount. In fact, the change can be shown to approach 0. In other words, the expression is approaching a limiting value. The limiting value of this expression is the **irrational** number e = 2.718281828459…, a non-terminating decimal.

By making the substitution $t = \frac{1}{x}$, we get the following alternate definition of Euler's number:

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{t \to 0} (1+t)^{\frac{1}{t}}$$

This limit is called the ____.

The Inverse of the Exponential Function, e^x

In previous mathematics courses, you learned that the inverse of an exponential function is the logarithmic function with the same base. Today, we will use similar ideas to graph of $g(x) = e^x$ *and* $y = \mathbf{log}_e\, \boldsymbol{x}$.

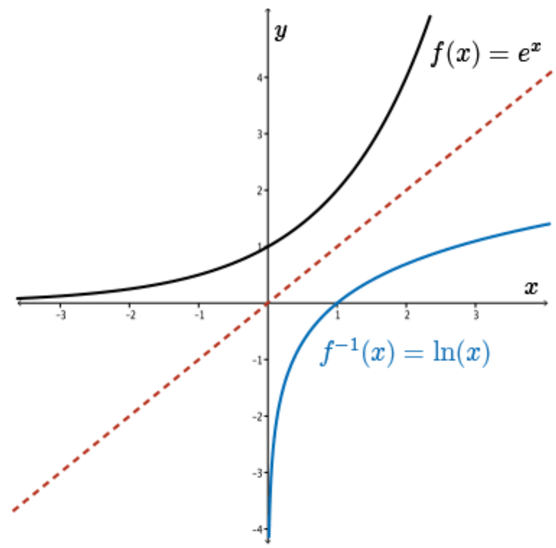
The inverse of the exponential function $g(x) = e^x$ is $g^{-1}(x) = \log_e(x) = \ln(x)$.

Rather than using $\log_e(x)$, mathematicians use $\boldsymbol{ln(x)}$ to shorten this expression. $\boldsymbol{ln(x)}$ stands for the natural logarithm of x and is pronounced "lawn x."

Like every function, we can find its inverse by reflecting the function in the line $y = x$. Hence, to graph $y = lnx$, we can interchange the values of x and y from the table of values of $y = e^x$.

x	$y = e^x$
-2	0.135
-1	0.36
0	1
1	2.72
2	7.39

x	$y = lnx$



	$y = e^x$	$y = lnx$
Domain		
Range		

In advanced functions, you learned the logarithmic laws and properties of logarithm. We can apply similar laws and properties when working with ln.

$\log_c ab = \log_c a + \log_c b$	$ln(ab) =$
$\log_c \frac{a}{b} = \log_c a - \log_c b$	$\ln\left(\frac{a}{b}\right) =$
$\log_c a^n = n \log_c a$	$ln(a^n) =$
$\log_a a = 1$	$ln\, e =$
$a^{\log_a x} = x$	$e^{lnx} =$
$\log_a 1 = 0$	$ln\, 1 =$

Example 1: Solve for x:

a) $5(10^{x+2}) = 200$

b) $10e^{2x-1} + 500 = 1000$

Example 2: Water is brought to a boil then removed from the heat. The temperature of the water, T degrees Celsius, is modeled as $T = 80e^{-0.57t} + 20$, where t is in minutes.

a) Determine the temperature after 15 minutes.

b) Determine how long it takes for the temperature to reach 30°C.

Example 3: A population of fish in a lake at time t months is given by the function $F(t)=\dfrac{20000}{1+24e^{-\frac{t}{4}}}$.

How long will it take for the fish population to reach 15 000?

The Derivative of $y=e^x$ and $y=\ln x$

The Derivative of f(x) = e^x

Proof: Let f(x) = e^x.

Then, from first principles: f'(x) =

If $f(x)=e^x$, then $f'(x)$ = ______________.

By the chain rule, if $f(x)=e^{g(x)}$, then $f'(x)$ = ______________________________.

Examples: Differentiate.

a) $f(x) = 3e^{x}$

b) $f(x) = 2e^{x^2}$

c) $f(x) = x^3 e^{-x}$

d) $f(x) = e^{x^2+3x}$

The Derivative of $y = ln\, x$

Proof:

If $y = ln\, x$ then y' =________________________.

Derivative of $y = \ln g(x)$: ______________________________.

Ex 1: Differentiate and simplify.

$a)$ $f(x) = x^2 \ln x$

$b)$ $f(x) = \dfrac{\ln x}{x}$

$c)$ $f(x) = \ln\left(\dfrac{x-1}{x+1}\right)$

$d)$ $f(x) = e^{x^2} \ln\left(\sqrt{x}\right)$

$e)$ $f(x) = \ln\left(\sqrt{x^3 + x^2}\right)$

Ex 2: Find the equation of the tangent line to the curve $y = e^{-x}$ that passes through the origin.

Ex 3: Find the points of inflection for $y = f(x) = e^{-x^2}$.

Ex 4: Find local extrema points for $f(x) = \frac{\ln x}{x}$.

Ex 5: Find the inflection points for $f(x) = x^2 \ln x$.

Ex 6: Find the local extrema for $y = f(x) = x^2 e^{-x^2}$.

Derivatives of $\boldsymbol{y = e^x}$ and $\boldsymbol{y = \ln(x)}$ Practice

Multiple Choice: *Identify the choice that best completes the statement or answers the question.*

____1. The graph of $y = e^x$ lies between the graphs of which two functions?

a. $y = 2x$ & $y = 3x$ b. $y = x^2$& $y = x^3$ c. $y = 3^x$ & 4^x d. $y = 2^x$ & 3^x

____2. Determine the value of x in the equation $lnx = 1$.

a. 10 b. 0 c. 1 d. e

____3. Simplify the expression $ln\, e^{3x}$.

a. $3\, ln\, e^{3x}$ b. $ln(3x)$ c. e^{3x} d. $3x$

____4. What is the value of $ln\ 0$?

a. 0 b. undefined c. 1 d. e

____5. Determine the value of x in the equation $2e^x = 6$.

a. $ln\ 6$ b. $ln\ 3$ c. 3 d. $\frac{ln6}{ln3}$

____ 6. If $f(x) = 2sinxcosx$, find $f'(\frac{\pi}{2})$.

a. –2 b. 1 c. $\frac{\pi}{2} + 1$ d. 2

____ 7. If $f(x) = cos(sinx)$, find $f'(1)$.

a. 0.2314 b. 1 c. 0 d. –0.4029

____ 8. What is the slope of the graph $y = 5e^x$ at $x = 1$?

a. e b. $5e$ c. 5 d. ln5

____ 9. If $f(x) = 2x^2e^x$, find $f'(1)$.

a. $2e$ b. $6e$ c. 0 d. $4e$

____ 10. If $f(x) = e^{x^2-x+1}$, find $f'(1)$.

a. 2 b. 1 c. 0 d. e

____ 11. If $f(x) = e^x\sqrt{x}$, find $f'(1)$.

a. $\frac{e}{2}$ b. $2e$ c. $\frac{3e}{2}$ d. e

____ 12. If $f(x) = \sqrt{x} + \sqrt{e^x} + \sqrt{e}$, find $f'(1)$.

a. $\frac{1+\sqrt{e}}{2}$ b. $\frac{1}{\sqrt{e}}$ c. $1 + e$ d. $\frac{1}{2}$

____ 13. If $f(x) = \sin(e^x)$, find $f'(0)$.

a. 1 b. 0 c. cos(1) d. cos(e)

____ 14. Find the point on the curve $y = e^x + x$ where the tangent is parallel to the line $y = 2x$.

a. $(1, 1 + e)$ b. (0, 1) c. $(-1, -1 + \frac{1}{e})$ d. $(-2, -2 + \frac{1}{e^2})$

____15.Find the point on the curve $y = e^{2x+1}$ where the tangent is parallel to the line $y = 2ex$.

a. $(1, e^3)$

b. $(0, e)$

c. $(\frac{1}{2}, e^2)$

d. $(-1, \frac{1}{e})$

Full Solutions:

16. Find the derivative. You do NOT have to simplify:

a. $f(x) = \ln(tan^2(sine^x))$ b. $f(x) = \mathrm{e}^{\sin x}\sec x$ c. $y = ln(ln\, 2x^4)$

17. Find the absolute maximum and absolute minimum of $f(x) = \frac{lnx}{x}$ on the closed interval $x\epsilon[2,10]$.

18. For $f(x) = x^4e^x$: Determine the intervals of increase and decrease.

19. Find y' (do not simplify): y = ($e^{\sqrt{x}} + xsecx + 1)^3$

20. Find y'' if $y = x^2e^{\sqrt{sin^2x+cos^2x+1}}$ ☺

21. Simplify the expression $(e^{2lnx})(ln\, e^{2x}) + ln\,(e^{x+1})$

22. Solve for x (to nearest hundredth if necessary):

a. $e^{-2x+3} = 4$ b. $\ln(x^2 - 2x) = 4$

Warm Up: Derivative of Other Functions

1. Differentiate each of the following.

(a) $f(x)=\ln\sqrt{\frac{x-1}{x+1}}+\frac{\ln[\cos(x)]}{\sin(x)}$ **(Do Not Simplify)**

(b) $y=e^{5\ln(x-3)}-e^{2}\sin^{3}(e^{x})$

2. Determine the exact point(s) of inflection of the function $f(x)=x(\ln(x))^2$

The Derivative of Exponential Functions $f(x) = b^x$

Proof:

The derivative of $y = b^x$ is: ___________________________.

The derivative of $y = b^{g(x)}$ is: ___________________________.

Ex 1: Differentiate.

a) $f(x) = 3^x$

b) $f(x) = \left(\frac{1}{2}\right)^x$

c) $f(x) = 3^{x^2}$

d) $f(x) = 4^{x^3+2x} \cdot 3^x$

Ex 2: Find the equation of the tangent line to the graph of $y = f(x) = x(2^{-x})$ at $(0, 0)$.

The Derivative of Logarithmic Functions $f(x) = \log_b(x)$

Proof:

The derivative of $y = \log_b(x)$ is: ________________________.

The derivative of $y = \log_b g(x)$ is: ________________________.

Ex 1: Differentiate.

a) $f(x) = x^2 \log_3 x$

b) $f(x) = \dfrac{\log x}{10^x}$

Ex 2: Differentiate.

a) $y = \log\sqrt{(x^2+1)}$

b) $f(x) = \log_2(x^2 2^x)$

c) $f(x) = \log(\ln x)$

Ex 3: Find the absolute extrema for $f(x) = \frac{\log x}{x}$ over the interval $[1, 10]$.

Ex 4. Determine the point on the graph of $f(x) = \ln(x + 4 + e^{-3x})$ where the tangent line is horizontal.

Ex 5: State the intervals of concavity for the function $f(x) = x\ln(4 - x^2)$.

<u>**The Derivative of Exponential and Logarithmic Functions Practice**</u>

Multiple Choice:

Identify the choice that best completes the statement or answers the question.

1. If $f(x) = 2^{sinx}$, find $f'(0)$

   a. 2 b. π c. $ln2$ d. 1

2. If $f(x) = 90(2^{\frac{x}{3}})$, find $f'(3)$

   a. $60ln2$ b. 30 c. $90ln2$ d. $180ln2$

3. If $f(x) = 2(3^{cosx})$, find $f'(\frac{\pi}{2})$.

   a. -2 b. 6 c. $-2ln3$ d. 2π

Full Solutions:

4. Find the derivative:

   a. $y = (10^{sinx})3^x$

   b. $y = sec^5(10^{e^x})$

   c. $y = \log_5 \frac{x^2+1}{x-1}$

   d. $y = \log_7 \sqrt{(x-1)(2x+1)(x+5)}$

   e. $y = \sec(\frac{x}{5^x})cos^2(\frac{x}{5^x})$

   f. $y = 5^x \, log \, e^{3x}$

5. Water is brought to a boil then removed from the heat. The temperature of the water, T degrees Celcius is modeled as $T = 80e^{-0.57t} + 20$. At what rate is the temperature decreasing when the temperature reaches 30°C.

6. The velocity of a car is given by $v(t) = 60[1-(0.7)^t]$, where v is measured in m/s and t is measured in seconds. Determine the time at which the acceleration is 3 m/s ^{2}.

7. The mass of polonium is given by the function, $M(t) = M_0\left(\frac{1}{2}\right)^{\frac{t}{138}}$ where M_0 is the initial mass of polonium, in milligrams, and $M(t)$ is the mass, in milligrams, after t days. At what rate is the polonium decaying when a 100 mg sample has decayed to 75% of its initial mass?

Optimization Problems with Exponential Functions

Ex. 1: The effectiveness of studying for a test depends on how many hours a student studies. Some experiments showed that if the effectiveness, E, is put on a scale of 0 to 10, then $E(t) = 0.5\left(10 + te^{\frac{-t}{20}}\right)$, where t is the number of hours spent studying for an examination. If a student has up to 30 h that he can spend studying, how many hours should he study for maximum effectiveness?

Ex 2: The number of insects on a small tropical island is given by $P(t) = 8000e^{0.03t}$ for the period starting January 1, 2001. *P(t)* represents the insect population after *t* months.

a) What was the population on January 1, 2001?

b) When would the population double? Round to 1 decimal place.(exact date)

c) Determine the rate of growth in population after 4 months.

Logarithmic Differentiation

Steps:

1. Take natural logarithms of both sides of an equation $y = f(x)$.
2. Simplify using logarithmic properties and laws.
3. Differentiate implicitly with respect to x.
4. Solve the resulting equation for y' or $\frac{dy}{dx}$. Note: you may need to substitute the expression for y into the derivative to express the derivative entirely in terms of x.

Ex. 1. Differentiate: $y = \dfrac{e^x \sqrt{x^2 - 1}}{(x+3)^2}$

Ex. 2. Find $\frac{dy}{dx}$ if $y=\sqrt{\frac{x^2+1}{x^2+2}}$, $y>0$ for all $x\in\Re$

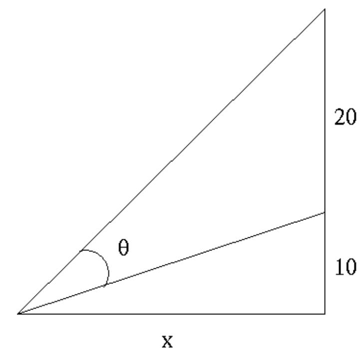
Ex. 3. Differentiate: $y=(sinx)^x$, $\sin x>0$.

Unit 5: Optimization for Exponential Functions-HW

1. Suppose that the monthly revenue in thousands of dollars, for the sale of x hundred units of an electronic item is given by the function $R(x) = 40x^2e^{-0.4x} + 30$, where the maximum capacity of the plant is 800 units. Determine the number of units to produce in order to maximize revenue.

2. The hypotenuse of aright triangle is 12 cm. Calculate the measure of the unknown angles in the triangle that will maximize the perimeter.

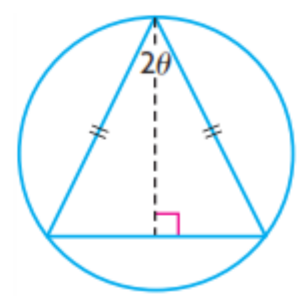
3. A movie screen on a wall is 20 meter high and 10 meter above the floor. At what distance x from the front of the room should you position yourself so that the tan ratio of viewing angle θ of the movie screen is as large as possible. (Hint: Maximize tan(θ)).



4. A drug is injected into the body in such a way that the concentration, C, in the blood at time t hours is given by the function $C(t) = 10(e^{-2t} - e^{-3t})$. At what time does the highest concentration occur within the first 5 h?

5. Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where the wire should stake to have the **angle** formed by the two pieces of wire at the stake be maximum?

6. An isosceles triangle is inscribed in a circle of radius *R*. Find the value of θ that maximizes the area of the triangle.



Warm Up

1. Differentiate-Do not simplify the answer:

a) $y = e^{x^2} \log_2\left[\sqrt[3]{4+x^2}\right]$

b) $f(x) = \left[\dfrac{\sec(x^2)}{1+\cos(x)}\right]^4$

2. Determine the slope of the normal line to the curve $\dfrac{x^2 + \sin(3y)}{x+y} = 3^{y^2}$ at point (1,0).

3. Using logarithmic differentiation, determine the derivative of $y = \left[\tan\left(4+x^2\right)\right]^{x+3}$.

Equations of Tangents and Normals

Recall: The normal to the graph of $y = f(x)$ at point P is the line that is **perpendicular** to the tangent at P. The slope of the normal is the **negative reciprocal** of the slope of the tangent.

Example 1: Determine the equation of the normal to the curve $y = 2xe^x$ at the point (0, 0).

Example 2: Find the equation of the tangent to the curve $f(x) = \ln(e^x + e^{2x})$ at the point where $x = 0$.

Example 3: A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle is given by $C(x) = 4\ln x + \left(\frac{30-x}{10}\right)^2$ with a minimum production capacity of 10 kettles per day. How many kettles should be manufactured to keep the cost per kettle to a minimum?

Example 4: Find the equation of the tangent line to the graph $f(x) = \sin^2 x$ at the point where $x = \frac{\pi}{3}$.

Practice: Sinusoidal, Exponential and Logarithmic Functions

1. Differentiate and simplify fully.

a) $f(t)=\sqrt{2+\sin^2 5t}$

b) $h(x)=\dfrac{x^2}{2-\cos \pi x}$

c) $f(t)=3e^{\sin 2t}$

d) $g(x)=e^{\sqrt{x}}\ln\sqrt{x}$

e) $y=2^x\log_2(x^4)$

f) $h(u)=\dfrac{e^{3u}}{1+e^{3u}}$

2. Find the maximum and minimum value of $f(x) = 2\sin x - x$, $0 \le x \le 2\pi$.

3. Find the maximum and minimum value of $f(x) = xe^{-2x}$ on the interval $0 \le x \le \ln 3$.

4. Determine the point(s) of inflection for $f(w) = \dfrac{\ln w^2}{w}$.

1. State in simplified form the derivative of the following:

   a) $y = \dfrac{\cos 2x}{\sin 2x\left(\sin^2 x - \cos^2 x\right)}$

   b) $y = \left(e^{-3\ln x}\right)^2 + \log_2 2^{\cos x^2}$

   c) $f(x) = \left(3x^2 + e\right) e^{-x}$

   d) $y = \dfrac{\sin\left(3x^2 + 1\right)}{\cos\left(3x^2 - 1\right)}$

2. Determine the local extrema of the function $f(x) = \ln\sqrt{x^3 - x}$.

3. The position of a particle is given by $s(t) = 5\sin\left(2t - \dfrac{\pi}{3}\right), 0 \le t \le 2\pi$, where ***t*** is in seconds and ***s*** is meters. Determine when the particle is at rest.

4. Find the point(s) on the graph of function $f(x) = (\ln x)^2 (\ln x - 1)$ that the tangent line is horizontal.

5. Find the values of a and b such that *f(x)* is differentiable everywhere

$$f(x) = \begin{cases} \dfrac{\sin x}{\sin x - \cos x} & , \quad x \le \dfrac{\pi}{2} \\ ax + b & , \quad x > \dfrac{\pi}{2} \end{cases}$$

6. Given $y = Axe^x + Be^x$, where A and B are constants ,show that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = -y$.

7. Determine the point(s) of inflection of the function $f(x) = \dfrac{1}{4}\cos(2x) + \sin(x)$ over the interval $x \in [0, 2\pi]$.

8. Determine the equation of the normal to the function $f(x) = 2x - (\ln x)^2$ at the point with *x*=e.

9. If $f(x) = e^{\cos(2x)}$ on [-π,π] determine and classify all local extrema.

10. For what value(s) of *k* does $y = e^{kx}\, sin(x)$ satisfy $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 0$?