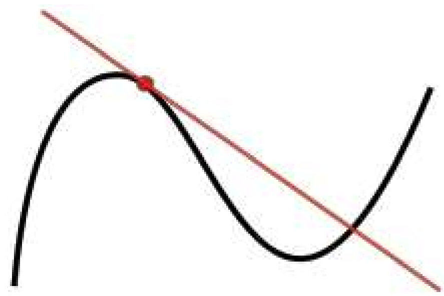
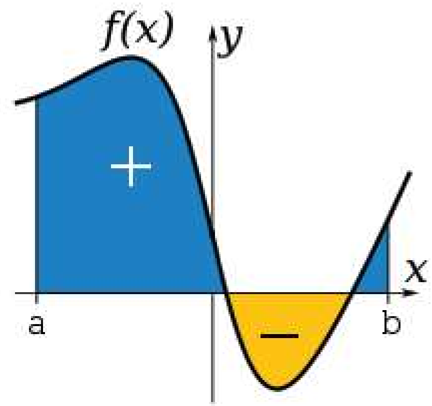
Introduction to Integration

Two main operations in calculus are differentiation and integration. Ironically, these operations can be thought of as being “opposite operations of each other”.

- When we **differentiate**, we find a **rate of change** at a particular point on a curve.
- When we **integrate**, we are attempting to find the **area** between the curve and the x-axis in some particular interval $[a, b]$

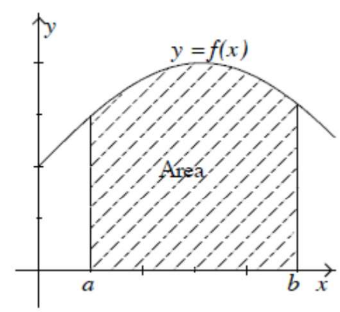




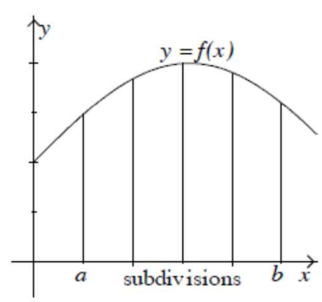
Warning: $\int_a^b f(x)dx$ measures the <u>difference in area</u> in the interval $[a, b]$ and NOT the area between the curve and the x-axis.

How did all of this come about?

Let us suppose that we are given a function $f(x)$ and we want to find the area enclosed between the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.



In order to do this, mathematicians decided to break up the area into **subdivisions**.

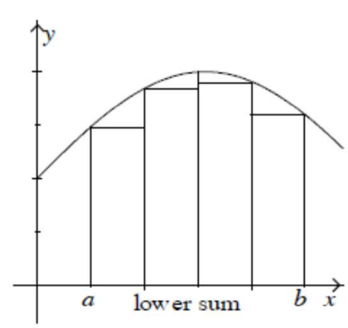


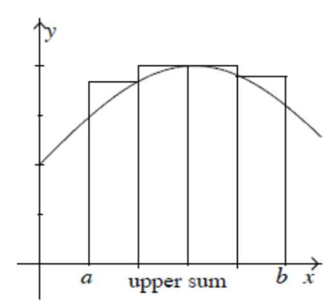
$$Length\ of\ subdivision = \frac{b-a}{\#\ of\ rectangles}$$

$$\Delta x = \frac{b-a}{n}$$

The area of each section can’t be calculated due curved top. In order to account for this, we can approximate the area by crafting rectangles.

- If we take the smaller rectangles we get the **lower sum**.
- If we take the larger rectangles we get the **upper sum**.



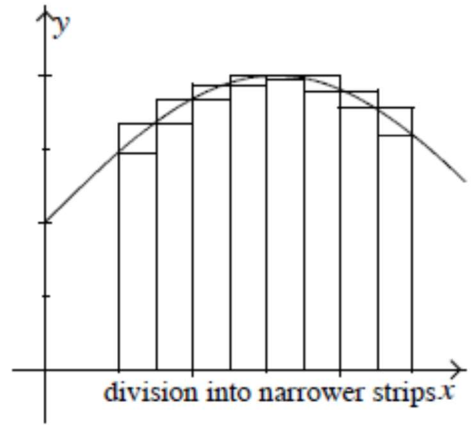


Since the lower sum will always give us an area **smaller** than what we want and the upper sum will give us an area always **larger** than what we want we get the following:

$$\sum all\ lower\ sum\ rectangles \leq Area\ Under\ curve \leq \sum all\ upper\ sum\ rectangles$$

$$\sum_{i=1}^{n} f(x_i^{\#})\Delta x \leq A \leq \sum_{i=1}^{n} f(x_i^{*})\Delta x \quad where\ n\ rep.\ the\ number\ of\ rectangles\ used.$$

Note: As the **number of rectangles used increases**, our approximations must get closer and closer to the **actual area**.



Therefore, our area in the interval $[a, b]$ must be given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

We will use a much simpler notation for this ☺

$$\boxed{A = \int_a^b f(x)dx}$$

Note: the $\int$ sign is an elongated 's' and stands for 'sum', just as the $\sum$ did previously. However, $\int$ **means <u>limit of a sum</u>** whereas $\sum$ **means <u>finite sum</u>**

<u>**A Crude Version of The Fundamental Theorem of Calculus**</u>

$$\frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

This means, if we take the derivative of an integral, <u>both operations cancel each other out</u> and we get back the original function.

- Note: The opposite is also true, if we integrate a derivative, we get back $f(x)$.

Moreover, if we want to solve for $\int f(x)dx$ then that means we need to move the $\frac{d}{dx}$ to the other side. Therefore, we must "**anti-differentiate $f(x)$**" to solve our integral.

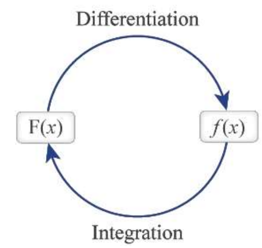
Suppose that $F(x)$ is the <u>**anti-derivative**</u> of $f(x)$**.** This means, if we take the derivative of $F(x)$ you get $f(x)$. If we anti-differentiate both sides of the above equation we end up getting

$$\boxed{\int f(x)dx = F(x) + c} \qquad where\ c\ is\ an\ unknown\ constant$$

- Note: $f(x)$ in the integral is called an **integrand**.

Integration

Anti-differentiation
The process of obtaining the original function from its derivative is called "**integration**' or "**anti-differentiation**'.



Example: If $f(x)=x^2$, then $f'(x)=2x$, so an anti-derivative (the opposite of a derivative) of $2x$ is x^2 or $F(x)=x^2$.

What about $f(x)=x^2+1$, $f(x)=x^2-\pi$, $f(x)=x^2+\frac{5}{2}$? $f'(x)=$ <u>$2x$</u>

In fact, there are many possible anti-derivatives of $2x$, each one differing from the others by a vertical translation. Hence, the anti-derivative of $2x$ is $F(x)=x^2+c$ where $c\in R$, and c is called the **constant of integration**.

A function $F(x)$ is an anti-derivative of $f(x)$ if $F'(x)=f(x)$ for all x in that interval.

Note: Always check to see if the anti-derivative is correct by differentiating the anti-derivative.

Example: If $f(x)=3x^2+5x-2$, then $F(x)=x^3+\frac{5x^2}{2}-2x+c$. Check: $F'(x)=$ <u>$3x^2+5x-2$</u>

Here are the anti-derivatives of some of the basic functions:

Anti-derivative $\qquad \frac{d}{dx}F(x)=f(x)$

$f(x)$	$F(x)$
1	$x+c$
$ax^n,\ n\neq -1$	$\frac{a}{n+1}x^{n+1}+c,\ n\neq -1$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)}+c,\ n\neq -1$
$x^{-1}=\frac{1}{x}$	$\ln\lvert x\rvert+c, n=-1$
$\frac{1}{ax+b} = (ax+b)^{-1}$	$\frac{\ln\lvert ax+b\rvert}{a}+c,\ ax+b>0$
ae^{kx}	$\frac{ae^{kx}}{k}+c$
$\sin(kx), k\neq 0$	$\frac{-\cos(kx)}{k}+c$
$\cos(kx), k\neq 0$	$\frac{\sin(kx)}{k}+c$

Ex: $\int 2\,dx = 2x+C$

Ex: $\int 2x^2\,dx = \frac{2x^3}{3}+C$

Ex: $\int (2x+1)^2\,dx = \frac{(2x+1)^3}{(2)(3)}+C$

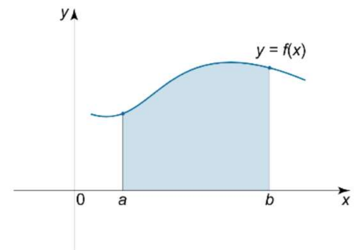
Two important rules for derivatives are

- The Constant Multiple Rule: $\int kf(x)dx = k\int f(x)dx$
- Sum or Difference Rule: $\int[f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Indefinite Integration

The set of all anti-derivatives of a function $f(x)$ is called the **indefinite integral of $f(x)$**, and is denoted by $\int f(x)dx$ where the symbol $\int$ is the integral sign like an elongated S indicating summation, and $f(x)$ is the integrand of the integral. $\therefore \int f(x)dx = F(x) + c$ where c is the constant of integration

Geometrical interpretation: Integration is a summation process of finding the area under the curve.



1. Find the indefinite integral of the following.

a) $\int (8x^2 + 4x - 3)dx$

$= 8\left(\frac{1}{3}x^3\right) + 4\left(\frac{1}{2}x^2\right) - 3x + C$

$= \frac{8}{3}x^3 + 2x^2 - 3x + C$

b) $\int [2\cos(2x)]dx$

$= \frac{2\sin(2x)}{2} + C$

$= \sin(2x) + C$

c) $\int \left(\frac{3}{x^2} - \frac{4}{x} + x\right)dx$

$= \int (3x^{-2} - 4x^{-1} + x)dx$

$= 3(-x^{-1}) - 4(\ln|x|) + \frac{1}{2}x^2 + C$

$= \frac{-3}{x} - 4\ln|x| + \frac{1}{2}x^2 + C$

d) $\int (e^x + 6e^{2x} - 8e^{-4x})dx$

$= e^x + \frac{6e^{2x}}{2} - \frac{8e^{-4x}}{-4} + C$

$= e^x + 3e^{2x} + 2e^{-4x} + C$

e) $\int \left(\frac{z^4 - 2z + 3}{z^2}\right)dz$

$= \int (z^2 - 2z^{-1} + 3z^{-2})dz$

$= \frac{z^3}{3} - 2\ln|z| + \frac{3z^{-1}}{-1} + C$

$= \frac{z^3}{3} - 2\ln|z| - \frac{3}{z} + C$

f) $\int \frac{\sin(x)}{\cos(x)}dx$

$= \int \left(\frac{1}{\cos(x)} \cdot \sin(x)\right) dx$

$= \frac{\ln|\cos(x)| \cdot \sin(x)}{-\sin(x)} + C$

$= -\ln|\cos(x)| + C$

Check: $\frac{d}{dx}(-\ln|\cos(x)|)$

$= -\frac{1}{\cos(x)} \cdot -\sin(x)$

g) $\int (3x+5)^3 dx$

$= \frac{(3x+5)^4}{4(3)} + C$

$= \frac{(3x+15)^4}{12} + C$

h) $\int \sin\left(\frac{1}{3}x - \frac{\pi}{4}\right)dx$

$= \frac{-\cos\left(\frac{1}{3}x - \frac{\pi}{4}\right)}{\frac{1}{3}} + C$

$= -3\cos\left(\frac{1}{3}x - \frac{\pi}{4}\right) + C$

1-1 Practice

Find the following indefinite integrals:

a) $\int x dx$

$= \frac{x^2}{2} + C$

b) $\int x^3 dx$

$= \frac{x^4}{4} + C$

c) $\int 4x^3 dx$

$= x^4 + C$

d) $\int 3x^2 + 6x + 5\, dx$

$= x^3 + 3x^2 + 5x + C$

e) $\int ax^n dx$

$= \frac{ax^{n+1}}{n+1} + C$

f) $\int 10\sqrt[3]{x^2}\ dx$

$= \frac{10x^{\frac{5}{3}}}{\frac{5}{3}} + C$

$= 6x^{\frac{5}{3}} + C$

g) $\int \frac{3x^3 dx}{4x^5}$

$= \frac{3}{4}\int x^{-2}\, dx$

$= \frac{-3}{4}x^{-1} + C$

$= -\frac{3}{4x} + C$

h) $\int \frac{5}{x} dx$

$= 5\ln|x| + C$

i) $\int \frac{x^5 - 3x + 7}{2x^2} dx$

$= \frac{1}{2}\int (x^3 - 3x^{-1} + 7x^{-2})\, dx$

$= \frac{x^4}{8} - \frac{3}{2}\ln|x| - \frac{7}{2x} + C$

j) $\int \frac{x^9 - 10}{\sqrt{x}}\ dx$

$= \int (x^{\frac{17}{2}} - 10x^{-\frac{1}{2}})\, dx$

$= \frac{2}{19}x^{\frac{19}{2}} - 20x^{\frac{1}{2}} + C$

k) $\int \frac{2}{5}e^x dx$

$= \frac{2}{5}e^x + C$

l) $\int 3e^{2x+5}\ dx$

$= \frac{3e^{2x+5}}{2} + C$

m) $\int 5\sin(x)\, dx$

$= -5\cos(x) + C$

n) $\int \frac{3}{4}\cos(x)\, dx$

$= \frac{3}{4}\sin(x) + C$

o) $\int (x^2 + 5)^2\ dx$

$= \int (x^4 + 10x^2 + 25)\, dx$

$= \frac{x^5}{5} + \frac{10x^3}{3} + 25x + C$

p) $\int (x + 3)^4\, dx$

$= \frac{(x+3)^5}{5} + C$

1-2 Warm Up

Find the indefinite integrals of the following:

1. $\int (6x-7)\,dx$

$= \frac{6x^2}{2} - 7x + C$

$= 3x^2 - 7x + C$

2. $\int \frac{1}{t^2}\,dt$

$= \frac{t^{-1}}{-1} + C$

$= -\frac{1}{t} + C$

3. $\int (x - \sqrt{x})\,dx$

$= \frac{x^2}{2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{1}{2}x^2 - \frac{2}{3}x^{\frac{3}{2}} + C$

4. $\int (x^3 - x^2 + 4x)\,dx$

$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{4x^2}{2} + C$

$= \frac{1}{4}x^4 - \frac{1}{3}x^3 + 2x^2 + C$

5. $\int \frac{2}{\sqrt[3]{x}}\,dx$

$= \int (2x^{-\frac{1}{3}})\,dx$

$= \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + C$

$= 3x^{\frac{2}{3}} + C$

6. $\int \frac{x^3+x^2+1}{x^3}\,dx$

$= \int (1 + x^{-1} + x^{-3})\,dx$

$= x + \ln|x| + \frac{x^{-2}}{-2} + C$

$= x + \ln|x| - \frac{1}{2x^2} + C$

7. $\int (8x + \cos(x))\,dx$

$= \frac{8x^2}{2} + \sin(x) + C$

$= 4x^2 + \sin(x) + C$

8. $\int (2e^x + \sin(x))\,dx$

$= 2e^x - \cos(x) + C$

Integrals with Initial Conditions
(Evaluation of the Constant of Integration)

A function has infinitely many anti-derivatives, differing only by an arbitrary constant, c. In applications, one is usually given additional information that helps to determine a value for the constant of integration, c, thereby specifying the unique anti-derivative that solves the problem. This information is often called a **boundary (or initial) condition**.

1. Find the displacement function s if $\frac{ds}{dt} = 4t$ when $s = 8$ at $t = 0$.

$$s = \int (4t)\,dt$$
$$= \frac{4t^2}{2} + c$$
$$= 2t^2 + c$$

At $t=0, s=8$:
$$8 = 2(0)^2 + c$$
$$8 = c$$

$\therefore$ the displacement function is $s = 2t^2 + 8$.

2. Given that $\frac{dy}{dx} = \sin(2x)$ and that $y = 1$ when $x = \frac{\pi}{6}$, find y as a function of x.

$$y = \int \sin(2x)\,dx$$
$$= \frac{-\cos(2x)}{2} + c$$

At $x = \frac{\pi}{6}, y = 1$:
$$1 = \frac{-\cos 2(\frac{\pi}{6})}{2} + c$$
$$1 = \frac{-\frac{1}{2}}{2} + c$$
$$\frac{5}{4} = c$$

$$\therefore y = -\frac{\cos(2x)}{2} + \frac{5}{4}$$

3. The gradient (slope) at any point on the curve $y = f(x)$ is given by the equation, $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}}$. The curve passes through the point (2, 3). Find the equation of this curve.

$$y = \int \frac{1}{\sqrt{x+2}}\,dx$$
$$= \int (x+2)^{-\frac{1}{2}}\,dx$$
$$= \frac{(x+2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= 2\sqrt{x+2} + c$$

At $(2,3)$:
$$3 = 2\sqrt{(2)+2} + c$$
$$3 = 4 + c$$
$$-1 = c$$

$\therefore$ the equation of this curve is $y = 2\sqrt{x+2} - 1$.

Application of Integration to Kinematics (Study of Motion)

Recall: $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$

In the context of integration, these ideas give $s(t) = \int v(t)dt$ and $v(t) = \int a(t)dt$.

Example 1: A particle is projected in a straight line relative to a fixed point, O with velocity function $v(t) = 25 - 10t,\ \ t \geq 0$. If the displacement of the particle from O at time $t = 2$ is 8, find:

a) the displacement function, $s(t)$ of the particle.
b) the displacement of the particle when $t = 4$.

a) $\int v(t) = s(t)$
$= \int (25 - 10t)dt$
$= 25t - \frac{10t^2}{2} + c$
$= 25t - 5t^2 + c$

$\because s(2) = 8,\ \therefore 8 = 25(2) - 5(2)^2 + c$
$8 = 50 - 20 + c$
$-22 = c$

$\therefore s(t) = -5t^2 + 25t - 22$ is the displacement function.

b) $s(4) = -5(4)^2 + 25(4) - 22$
$= -80 + 100 - 22$
$= -2$

$\therefore$ the displacement is -2 units.

Example 2: The acceleration, in m/s^2 of a body in a medium is given by $\frac{dv}{dt} = \frac{3}{t+1}, t \geq 0$. The particle has an initial speed of 6 m/s. Find the speed after 10 s.

$v = \int \left(\frac{3}{t+1}\right) dt$
$= 3\int (t+1)^{-1} dt$
$= 3\ln|t+1| + c$

At $t=0, v=6$:
$(6) = 3\ln|(0)+1| + c$
$6 = c$

$\therefore v = 3\ln|t+1| + 6$

At $t = 10$s,
$v = 3\ln|(10)+1| + 6$
$v \doteq 13.2$

$\therefore$ the speed after 10s is approx. 13.2 m/s.

SOLUTIONS

Integrals with Initial Conditions Extra Practice

1. Given that $f'(x) = x^2 e^{x^3}$ and $f(2) = 5e^2$, determine $f(x)$. $\left[f(x) = \frac{e^{x^3}}{3} + 5e^2 - \frac{e^8}{3}\right]$

$$f(x) = \int x^2 e^{x^3} dx$$
$$= \frac{e^{x^3}}{3x^2} x^2 + C$$
$$= \frac{e^{x^3}}{3} + C$$

$\because f(2) = 5e^2$

$$5e^2 = \frac{e^{(2)^3}}{3} + C$$
$$5e^2 - \frac{e^8}{3} = C$$
$$\therefore f(x) = \frac{e^{x^3}}{3} + 5e^2 - \frac{e^8}{3}$$

2. Find $f(x)$ if $f''(x) = x^{-\frac{1}{2}} + x^{-\frac{2}{3}}$, $f(0) = 5$, and $f(1) = 0$. $\left[f(x) = \frac{4}{3}x^{\frac{3}{2}} + \frac{9}{4}x^{\frac{4}{3}} - \frac{103}{12}x + 5\right]$

$$f'(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + C_1$$
$$= 2x^{\frac{1}{2}} + 3x^{\frac{1}{3}} + C_1$$

$$f(x) = \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{4}{3}}}{\frac{4}{3}} + C_1 x + C_2$$
$$= \frac{4}{3}x^{\frac{3}{2}} + \frac{9}{4}x^{\frac{4}{3}} + C_1 x + C_2$$

$f(0)=5$: $5 = \frac{4}{3}(0)^{\frac{3}{2}} + \frac{9}{4}(0)^{\frac{4}{3}} + C_1(0) + C_2$

$$5 = C_2$$

$f(1)=0$: $0 = \frac{4}{3}(1)^{\frac{3}{2}} + \frac{9}{4}(1)^{\frac{4}{3}} + C_1(1) + (5)$

$$0 = \frac{4}{3} + \frac{9}{4} + C_1 + 5$$
$$-\frac{103}{12} = C_1$$

$$\therefore f(x) = \frac{4}{3}x^{\frac{3}{2}} + \frac{9}{4}x^{\frac{4}{3}} - \frac{103}{12}x + 5$$

3. A fighter jet begins a downward descent in order to pick up some vertical speed. Its acceleration can be modelled by the equation $a = 0.48t^2 + 2$ for $0 < t < 10$, where a is measured in $\frac{m}{s^2}$ and t is the time in seconds from where the fighter jet begins his descent. If his velocity at t = 2 is approximately $8\,\frac{m}{s}$, determine the velocity after 8 seconds have passed. [100.64 m/s]

$$v = \int (0.48t^2 + 2)\, dt$$
$$= \frac{0.48t^3}{3} + 2t + C$$
$$= 0.16t^3 + 2t + C$$

sub in $v=8$, $t=2$:

$$8 = 0.16(2)^3 + 2(2) + C$$
$$2.72 = C$$

$$\therefore v = 0.16t^3 + 2t + 2.72$$

sub $t=8$,

$$v = 0.16(8)^3 + 2(8) + 2.72$$
$$= 100.64$$

$\therefore$ The velocity after 8 seconds have passed is 100.64 m/s.

Extra Problems

1. $\int x(x^2+6)^5 dx$

2. $\int (2x-1)\sqrt[5]{(x^2-x+4)^3}\, dx$

3. $\int \frac{3x^3}{(x^4+5)^2} dx$

4. $\int x^3\sqrt{x^2+6}\, dx$

5. $\int \sec^2(x)\, dx$

6. $\int \sin^7(x)\cos(x)\, dx$

7. $\int \frac{2\sin(x)\cos(x)}{1-2\cos^2(x)} dx$

8. $\int \frac{2x}{x^2+3} dx$

9. $\int (3x^2-2x)e^{x^3-x^2+4} dx$

10. $\int 2\sec^2(x) - \sec(x)\tan(x)\, dx$

11. $\int \tan(x)\, dx$

12. Find y if $\frac{dy}{dx} = 3\sin(2x)$ and $\left(\frac{\pi}{2}, 0\right)$ is on the curve.

13. Find $f(x)$ given that $f''(x) = 2 - \frac{2}{\sqrt{x^3}}$ where $f'(1) = 0$, and $f(1) = 8$

Answers

1. $\frac{1}{12}(x^2+6)^6 + c$
2. $\frac{5}{8}(x^2-x+4)^{\frac{8}{5}} + c$
3. $-\frac{3}{4(x^4+5)} + c$
4. $\frac{1}{5}(x^4-4)(x^2+6)^{\frac{3}{2}} + c$
5. $\tan(x) + c$
6. $\frac{\sin^8(x)}{8} + c$
7. $\frac{1}{2}\ln(\cos(2x)) + c$
8. $\ln(x^2+3) + c$
9. $e^{x^3-x^2+4} + c$
10. $2\tan(x) - \sec(x) + c$
11. $-\ln(\cos(x)) + c$
12. $y = -\frac{3}{2}\cos(2x) - \frac{3}{2}$
13. $f(x) = x^2 + 8\sqrt{x} - 6x + 5$

4 alternate answer: $\frac{1}{5}(x^2+6)^{\frac{5}{2}} - 2(x^2+6)^{\frac{3}{2}} + c$

Substitution Rule

In general, the rule is used to find the anti-derivative of a function using chain rule.
If $u = g(x)$, then $\int f(g(x))g'(x)dx = \int f(u)du$.

Steps in integration by substitution (change of variable) rule:

1. Define u (i.e. let u be a function of the variable (often x) which is part of the integrand)
2. Convert the integrand from an expression of x to an expression in u (need to convert the "dx" term to a "du" term)
3. Integrate and then rewrite answer in terms of x (substitute back for u)

Example: Find the following indefinite integrals.

a) $\int (2x+1)^4 dx \rightarrow \int \frac{u^4}{2} \cdot du$

Let $u=2x+1$, $\therefore \frac{du}{dx}=2$, $\frac{du}{2}=dx$

$= \frac{u^5}{10}+C$

$= \frac{(2x+1)^5}{10}+C$

b) $\int 2x(x^2+1)^3 dx \rightarrow \int u^3 du$

Let $u=x^2+1$, $\frac{du}{dx}=2x$, $du=2x\,dx$

$= \frac{u^4}{4}+C$

$= \frac{(x^2+1)^4}{4}+C$

c) $\int \frac{x^2}{\sqrt{x^3-4}} dx \rightarrow \int \frac{u^{-\frac{1}{2}}}{3} du$

Let $u=x^3-4$, $du=3x^2dx$, $\frac{du}{3}=x^2dx$

$= \frac{u^{\frac{1}{2}}}{3\left(\frac{1}{2}\right)}+C$

$= \frac{2}{3}u^{\frac{1}{2}}+C$

$= \frac{2}{3}(x^3-4)^{\frac{1}{2}}+C$

d) $\int x\sqrt{x-1}\,dx \rightarrow \int (u+1)(u^{\frac{1}{2}})\,du$

Let $u=x-1$, $\frac{du}{dx}=1$, $du=dx$

Also, $x=u+1$

$= \int \left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du$

$= \frac{2u^{\frac{5}{2}}}{5}+\frac{2u^{\frac{3}{2}}}{3}+C$

$= \frac{2(x-1)^{\frac{5}{2}}}{5}+\frac{2(x-1)^{\frac{3}{2}}}{3}+C$

e) $\int \frac{3\ln(x)}{x} dx \rightarrow 3\int u \cdot du$

Let $u=\ln(x)$, $\frac{du}{dx}=\frac{1}{x}$, $du=\frac{1}{x}\cdot dx$

$= 3 \cdot \frac{u^2}{2}+C$

$= \frac{3(\ln|x|)^2}{2}+C$

f) $\int (4+\sin(x))^9\cos(x)\,dx \rightarrow \int u^9 \cdot du$

Let $u=4+\sin(x)$, $\frac{du}{dx}=\cos(x)$, $du=\cos(x)\cdot dx$

$= \frac{u^{10}}{10}+C$

$= \frac{(4+\sin(x))^{10}}{10}+C$

g) $\int \frac{3x^2+1}{x^3+x+5} dx \rightarrow \int u^{-1} du$

Let $u=x^3+x+5$, $\frac{du}{dx}=3x^2+1$, $du=(3x^2+1)dx$

$= \ln|u|+C$

$= \ln|x^3+x+5|+C$

h) $\int xe^{x^2+1} dx \rightarrow \int \frac{e^u}{2}\cdot du$

Let $u=x^2+1$, $\frac{du}{dx}=2x$, $\frac{du}{2}=x\cdot dx$

$= \frac{e^u}{2}+C$

$= \frac{e^{x^2+1}}{2}+C$

Variable Acceleration and Applications Extra Problems

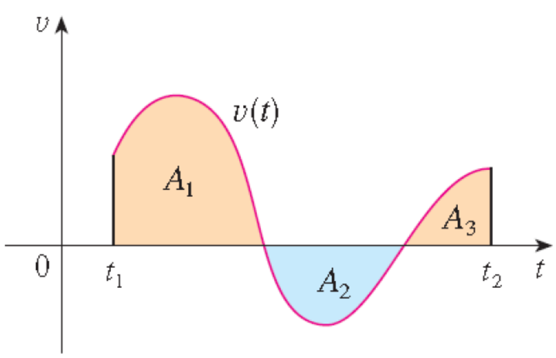
1. The velocity of a car moving in heavy traffic is given by $v(t) = 30t + 20\sin(10\pi t)$ where velocity is in km/hr starting at $t = 0$. How far does the car travel in the first 30 min of traffic? $[\frac{15}{4}+\frac{4}{\pi}\text{ km}]$

2. A model rocket is fired vertically upward. When its fuel is exhausted, it has a velocity of $30\frac{m}{s}$ and has reached a height of $80\ m$. Assume that the acceleration due to gravity is $-10\ m/s^2$ and neglect the effects of air resistance.
   a) For how long will the rocket continue to rise? $[3\ s]$
   b) Find the maximum height attained. $[125\ m]$
   c) When will the rocket strike the ground? $[8\ s]$

3. A jet propelled sled starts a run at time $t = 0$, with an initial velocity of $11\ m/s$. Its acceleration at time $t \geq 0$ is given by $a(t) = 10 - 2t$.
   a) When does the sled come to rest? $[11\ s]$
   b) How far will the sled have travelled at this time? $\left[\frac{847}{3}m\right]$

4. The displacement of an object in meters varies with time in seconds as $s(t) = -\frac{1}{3}t^3 + \frac{3}{2}t^2 + 4t$ for $0 \leq t \leq 5$. Find the maximum velocity of the object. $\left[6.25\frac{m}{s}\right]$

<u>Challenge Problem (do this after tomorrow's lesson)</u>



$$\text{displacement} = \int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3$$

$$\text{distance} = \int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3$$

5. A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).
   a) Find the displacement of the particle during the time period $1 \leq t \leq 4$. [4.5 m to the left]
   b) Find the distance traveled during the time period in a) $[10.17\ m]$

<u>Variable Acceleration and Applications Extra Problems-Solutions</u>

1.

$s(t) = \int [30t + 20\sin(10\pi t)]\,dt$

$= 15t^2 + 20\left(-\dfrac{\cos(10\pi t)}{10\pi}\right) + c$

$\because v(0) = 0$

$(0,0)$ must lie on $s(t)$

$0 = -\frac{2}{\pi}\cos(0) + c$

$c = \frac{2}{\pi}$

$s(0.5) = 15(0.5)^2 - \frac{2}{\pi}\cos(5\pi) + \frac{2}{\pi}$

$= 3.75 + \frac{4}{\pi}$

2. a) $a(t) = -10$

$\because$ fuel is exhausted $v(0) = 30$

$v(t) = -10t + c_1$

$30 = -10(0) + c_1$

$c_1 = 30$

$s(t) = -5t^2 + c_1 t + c_2$

$s(0) = 80$

$80 = -5(0)^2 + 30(0) + c_2$

$c_2 = 80$

$\therefore s(t) = -5t^2 + 30t + 80$

$0 = -10t + 30$

$10t = 30$

$t = 3s.$

$\therefore$ it will continue to rise for 3s. (vertex of $s(t)$)

b) $s(3) = -5(3)^2 + 30(3) + 80$

$= 125\text{ m}.$

c) $s(t) = -5(t^2 - 6t - 16)$

$0 = -5(t-8)(t+2)$

$\therefore t = 8$ or $\cancel{t = -2}$. inadmissible

$\therefore t = 8s.$

3. a) $v(t) = \int (10-2t)\,dt$

$= 10t - t^2 + c$

$11 = 10(0) - (0)^2 + c$

$c = 11$

$\therefore\ v(t) = -t^2 + 10t + 11$

$0 = -(t^2 - 10t - 11)$

$= -(t-11)(t+1)$

$\therefore t = 11$ or $t \cancel{= -1}$

$\therefore$ comes to rest $t = 11$s.

b) $s(t) = \int (-t^2 + 10t + 11)\,dt$

$= -\frac{t^3}{3} + 5t^2 + 11t + c$

starts run @ $t = 0$, $\therefore (0,0)$

$0 = c$

$s(11) = -\frac{(11)^3}{3} + 5(11)^2 + 11(11)$

$\doteq 282$ or $\frac{847}{3}$ m

4. $v(t) = -t^2 + 3t + 4$

$a(t) = -2t + 3$

$0 = -2t + 3$

$2t = 3$

$t = \frac{3}{2}$

$v(\frac{3}{2}) = -(\frac{3}{2})^2 + 3(\frac{3}{2}) + 4$

$= 6.25$ m/s

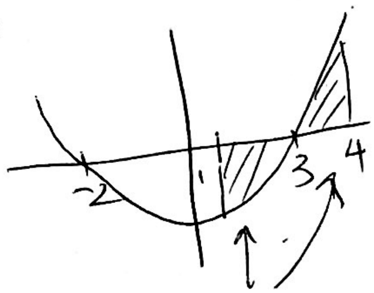
5 a) $s(t) = \int_1^4 (t^2 - t - 6)\,dt$

$$= \frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t \Big]_1^4$$

$$= \left[\frac{1}{3}(4)^3 - \frac{1}{2}(4)^2 - 6(4)\right] - \left[\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 - 6(1)\right]$$

$= -4.5\text{m}$ or 4.5m left of starting point.

b)



distance is total area

$$t^2 - t - 6 = 0$$
$$(t+2)(t-3) = 0$$
$$t = -2, 3.$$

abs value since area above curve

$$d = \left|\int_1^3 (t^2 - t - 6)\,dt\right| + \int_3^4 (t^2 - t - 6)\,dt.$$

$\vdots$

$= 10.17\text{m}.$

OR $d = \int_1^4 |t^2 - t - 6|\,dt.$

$\doteq 10.17\text{m}.$

1-3 Warm Up

1. Find the indefinite integral:

a) $\int (x^2-5)^8 2x\,dx$

Let $u = x^2-5$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$= \int u^8\,du$

$= \frac{u^9}{9} + C$

$= \frac{1}{9}(x^2-5)^9 + C$

b) $\int \frac{x^2}{\sqrt{1-x^3}}dx \rightarrow \int \frac{u^{-\frac{1}{2}}}{-3}\,du$

Let $u = 1-x^3$

$\frac{du}{dx} = -3x^2$

$\frac{du}{-3} = x^2\,dx$

$= -\frac{1}{3}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= -\frac{2}{3}(1-x^3)^{\frac{1}{2}} + C$

c) $\int \sin(4x)\,dx$

$= \frac{-1}{4}\cos 4x + C$

d) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx \rightarrow \int e^u \cdot 2\,du$

Let $u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$2\,du = \frac{dx}{\sqrt{x}}$

$= 2e^u + C$

$= 2e^{\sqrt{x}} + C$

Challenge questions!

2. Find the indefinite integral:

a) $\int \sin^2(x)\,dx$

Note:
$\cos(2x) = 1 - 2\sin^2(x)$
$\frac{\cos(2x)-1}{-2} = \sin^2(x)$

$= \int \frac{\cos(2x)-1}{-2}\,dx$

$= -\frac{1}{2}\int (\cos(2x)-1)\,dx$

$= -\frac{1}{2}\left(\frac{\sin(2x)}{2} - x\right) + C$

$= \frac{-1}{4}\sin(2x) + \frac{1}{2}x + C$

b) $\int \frac{\cos(2x)+1}{\cos(x)}dx$

Note:
$\cos(2x) = 2\cos^2(x) - 1$

$= \int \frac{(2\cos^2(x)-1)+1}{\cos(x)}\,dx$

$= \int (2\cos(x))\,dx$

$= 2\sin(x) + C$

c) $\int x^3\sqrt{x^2-6}\,dx$

Let $u = x^2-6$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$

$u+6 = x^2$

$= \int x \cdot x^2\sqrt{x^2-6}\,dx$

$= \int (u+6)\sqrt{u}\,\frac{du}{2}$

$= \frac{1}{2}\int \left(u^{\frac{3}{2}} + 6u^{\frac{1}{2}}\right)du$

$= \frac{1}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + C$

$= \frac{1}{5}(x^2-6)^{\frac{5}{2}} + 2(x^2-6)^{\frac{3}{2}} + C$

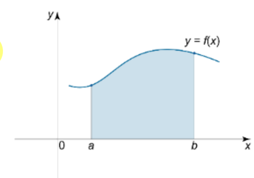
Definite Integral

If the function, $f(x)$, is continuous at every point on the interval $[a, b]$ and $F(x)$ is any anti-derivative of $f(x)$ on $[a, b]$, then $\int_a^b f(x)dx$ is called the **definite integral** and is equal to $F(b) - F(a)$.

$\therefore \boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)}$, where a and b are the lower and upper limits of integration respectively.

Note:

- c is omitted as it would cancel itself out as shown: $[F(b) + c] - [F(a) + c] = F(b) - F(a)$
- a definite integral has a numerical value
- the evaluation of the definite integral $A = \int_a^b f(x)dx$ represents the area enclosed between the graph of $f(x)$, the x-axis and the lines $x = a$ and $x = b$.



Properties of the Definite Integral

If f and g are continuous functions defined on a given interval $[a, b]$ and c is a constant, then:

1) $\int_a^a f(x)dx = 0$

2) $\int_a^b cdx = c(b - a)$

3) $\int_b^a cf(x)dx = c\int_a^b f(x)dx$

4) $\int_b^a f(x)dx = -\int_a^b f(x)dx$

5) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

6) $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

Example 1. Evaluate.

a) $\int_3^5 \frac{1}{x}dx$

$= \ln|x|\Big]_3^5$ or $\ln|x|\Big|_3^5$

$= \ln(5) - \ln(3)$

$\doteq 0.511$

b) $\int_2^4 \left(x + \frac{1}{x}\right)^2 dx$

$= \int_2^4 \left(x^2 + 2 + \frac{1}{x^2}\right)dx$

$= \frac{x^3}{3} + 2x - \frac{1}{x}\Big]_2^4$

$= \left[\frac{(4)^3}{3} + 2(4) - \frac{1}{(4)}\right] - \left[\frac{(2)^3}{3} + 2(2) - \frac{1}{(2)}\right]$

$= \frac{275}{12} \doteq 22.9$

c) $\int_0^1 \left(e^{2x} + \frac{3}{x+1}\right)dx$

$= \frac{e^{2x}}{2} + 3\ln|x+1|\Big]_0^1$

$= \left[\frac{e^{2(1)}}{2} + 3\ln|(1)+1|\right] - \left[\frac{e^{2(0)}}{2} + 3\ln|(0)+1|\right]$

$= \frac{e^2}{2} + 3\ln|2| - \frac{1}{2}$

$\doteq 5.27$

d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin(3x)dx$

$= \frac{-\cos(3x)}{3}\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= -\frac{1}{3}\left[\cos 3\left(\frac{\pi}{2}\right) - \cos 3\left(\frac{\pi}{6}\right)\right]$

$= -\frac{1}{3}(0-0)$

$= 0$

e) $\int_5^2 (3x-4)^4 dx$

$= \frac{(3x-4)^5}{3(5)}\Big]_5^2$

$= \frac{(3x-4)^5}{15}\Big]_5^2$

$= \frac{1}{15}\left[[3(2)-4]^5 - [3(5)-4]^5\right]$

$= \frac{1}{15}\left(2^5 - 11^5\right)$

$= \frac{-161\ 019}{15}$

or $\doteq -10\ 734.6$

f) $\int_{-2}^0 (x - e^{-x})dx$

$= \frac{x^2}{2} + e^{-x}\Big]_{-2}^0$

$= \left[\frac{(0)^2}{2} + e^{-(0)}\right] - \left[\frac{(-2)^2}{2} + e^{-(-2)}\right]$

$= 1 - 2 - e^2$

$= -1 - e^2$

or $\doteq -8.39$

g) $\int_0^1 x^4(x^5+1)^5\,dx \rightarrow = \int_1^2 \frac{u^5}{5}\,du$

Let $u = x^5+1$,

$\frac{du}{dx} = 5x^4$

$\frac{du}{5} = x^4\,dx$

when $x=0, u=1$

when $x=1, u=2$

$= \frac{u^6}{30}\Big]_1^2$

$= \frac{(2)^6}{30} - \frac{(1)^6}{30}$

$= \frac{21}{10}$

h) $\int_0^5 (3x - x^2)\,dx$

$= \frac{3x^2}{2} - \frac{x^3}{3}\Big]_0^5$

$= \left[\frac{3(5)^2}{2} - \frac{(5)^3}{3}\right] - \left[\frac{3(0)^2}{2} - \frac{(0)^3}{3}\right]$

$= \frac{75}{2} - \frac{125}{3}$

$= \frac{-25}{6}$ or $\doteq -4.17$

2. Find the area under the curve $y = \frac{2}{2x+1}$ from $x = 0$ to $x = 1$.

Area $= \int_0^1 \frac{2}{2x+1}\,dx$

$= \frac{2\ln|2x+1|}{2}\Big]_0^1$

$= \ln|2(1)+1| - \ln|2(0)+1|$

$= \ln(3) - \ln(1)$

$= \ln(3)$

OR

Let $u = 2x+1$

$\frac{du}{dx} = 2$

$du = 2\,dx$

when $x=0, u=1$

when $x=1, u=3$

Area $= \int_0^1 \frac{2}{2x+1}\,dx$

$= \int_1^3 \frac{1}{u}\,du$

$= \ln|u|\Big]_1^3$

$= \ln(3) - \ln(1)$

$= \ln(3)$ or $\doteq 1.10$

$\therefore$ the area is $\ln(3)$ units^2.

1. A curve with equation y= f(x) passes through the point (1, 0).The gradient of the curve, $f'(x)=\dfrac{a(\ln x)^4+b}{x}$, at point $\left(e,\dfrac{4}{5}\right)$ is $\dfrac{1}{e}$. Find the values of a and b.

2. Find the area of the region between the curve $y=4e^{2x}-3e^{x}$ and the x-axis from x=1 to x=2.

3. A particle moves such that its acceleration, a ms^{-2}, at time t seconds is given by formula $a(t)=\dfrac{2}{\sqrt{t}}+10t$, $t>0$. At 4 seconds, the velocity of the particle is 28 ms^{-1}.

   a) Find an expression for the velocity of particle at time t.

   b) Hence, find the velocity of particle at $\frac{1}{4}$ seconds.

4. Determine the value of k > 1 if $\displaystyle\int_1^k\left(1+\frac{1}{x^2}\right)dx=\frac{3}{2}$.

MCV4UE Warm-Up

slope

1. A curve with equation y= f(x) passes through the point (1, 0).The gradient of the curve, $f'(x)=\dfrac{a(\ln x)^4+b}{x}$, at point $\left(e,\dfrac{4}{5}\right)$ is $\dfrac{1}{e}$. Find the values of a and b.

$$f'(e)=\frac{1}{e}$$

$$\frac{a(\ln e)^4+b}{e}=\frac{1}{e}$$

$$a+b=1 \quad (1)$$

$$f(x)=a\int\frac{(\ln x)^4}{x}dx+b\int\frac{1}{x}dx$$

$$\ln x=u \uparrow \frac{dx}{x}=du$$

$$f(x)=a\int u^4du+b\int\frac{1}{x}dx$$

$$=\frac{a}{5}(\ln x)^5+b\ln|x|+c$$

$$f(1)=0 \uparrow c=0$$

$$f(e)=\frac{4}{5} \uparrow \frac{a}{5}+b=\frac{4}{5}$$

$$a+5b=4 \quad (2)$$

$$(2)-(1): 4b=3$$

$$b=\frac{3}{4} \ \& \ a=\frac{1}{4}$$

Full solutions : $f'(e)=\frac{1}{e}$, $\quad \frac{1}{e}=\frac{a[\ln(e)]^4+b}{(e)}$

$1=a(1)^4+b$

$1=a+b \rightarrow ①$

$f(x)=\int f'(x)\,dx$

$=\int\frac{a[\ln(x)]^4+b}{x}dx$

$=a\int\frac{[\ln(x)]^4}{x}+b\int\frac{1}{x}dx \quad \rightarrow$ let $u=\ln(x)$, $\frac{du}{dx}=\frac{1}{x}$, $du=\frac{1}{x}dx$

$=a\int u^4du+b\int\frac{1}{x}dx$

$=a\left(\frac{u^5}{5}\right)+b\ln|x|+C$

$f(x)=\frac{a}{5}[\ln(x)]^5+b\ln(x)+C$

$\because f(1)=0 \rightarrow 0=\frac{a}{5}[\ln(1)]^5+b\ln(1)+C$

$0=C \quad \therefore f(x)=\frac{a}{5}[\ln(x)]^5+b\ln(x)$

$\because f(e)=\frac{4}{5} \rightarrow \frac{4}{5}=\frac{a}{5}[\ln(e)]^5+b\ln(e)$

$\frac{4}{5}=\frac{a}{5}+b$

$4=a+5b \rightarrow ②$

Elimination : ② - ① : $4b=3$

$b=\frac{3}{4}$

sub $b=\frac{3}{4}$ into ① : $a+\left(\frac{3}{4}\right)=1$

$a=\frac{1}{4}$

2. Find the area of the region between the curve $y=4e^{2x}-3e^{x}$ and the x-axis from x=1 to x=2. Check your answer using GDC.

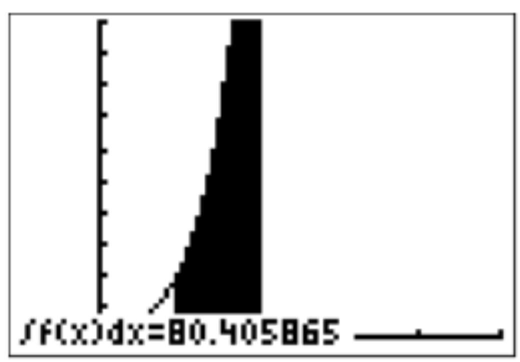
$$A=\int_1^2\left(4e^{2x}-3e^{x}\right)dx$$

$$=2e^{2x}-3e^{x}\Big]_1^2$$

$$=\left(2e^4-3e^2\right)-\left(2e^2-3e\right)$$

$$=2e^4-5e^2+3e$$

$$\doteq 80.406\ u^2$$



3. A particle moves such that its acceleration, a ms^{-2}, at time t seconds is given by formula $a(t) = \frac{2}{\sqrt{t}} + 10t$, $t > 0$. At 4 seconds, the velocity of the particle is 28 ms^{-1}.

a) Find an expression for the velocity of particle at time t.

$$v(t) = \int\left(\frac{2}{\sqrt{t}} + 10t\right)dt$$
$$= 4\sqrt{t} + 5t^2 + c$$
$$28 = 4\sqrt{4} + 5(4)^2 + c$$
$$28 - 88 = c$$
$$c = -60$$
$$v(t) = 4\sqrt{t} + 5t^2 - 60$$

b) Hence, find the velocity of particle at $\frac{1}{4}$ seconds.

$$v(4) = 4\sqrt{\frac{1}{4}} + 5(\frac{1}{4})^2 - 60$$
$$= 2 + \frac{5}{16} - 60$$
$$= -\frac{923}{16}$$
$$\doteq -57.7 m/s$$

4. Determine the value of $k > 1$ if $\int_1^k \left(1 + \frac{1}{x^2}\right)dx = \frac{3}{2}$.

$$\int_1^k \left(1 + \frac{1}{x^2}\right)dx = \frac{3}{2}$$
$$\left. x - \frac{1}{x}\right]_1^k = \frac{3}{2}$$
$$k - \frac{1}{k} = \frac{3}{2}$$
$$2k^2 - 3k - 2 = 0$$
$$(2k+1)(k-2) = 0$$
$$k = \frac{-1}{2}, \quad \boxed{k = 2}$$

not in domain

1-4 Warm Up

1. Find the equation of the curve with $\frac{dy}{dx} = x^2 + 6x - 3$ and passes through the point (3, 10).

2. Find the equation of the curve $y = F(x)$ that passes through the point (1, 4) and satisfies $\frac{dy}{dx} = 9x^2 - e^{2x} + 1$.

3. Determine the function, $f(x)$, if $f'(x) = \sin(2x) - \cos\left(\frac{x}{3}\right)$ and it passes through the point $\left(\frac{\pi}{2}, -3\right)$.

Solutions:

① $y = \int (x^2 + 6x - 3)\,dx$

$= \frac{1}{3}x^3 + \frac{1}{2}(6x^2) - 3x + C$

$= \frac{1}{3}x^3 + 3x^2 - 3x + C$

At $x = 3$, $y = 10$,

$10 = \frac{1}{3}(3)^3 + 3(3)^2 - 3(3) + C$

$10 = 27 + C$

$-17 = C$

$\therefore$ The equation of the curve is $y = \frac{1}{3}x^3 + 3x^2 - 3x - 17$.

② $F(x) = \int (9x^2 - e^{2x} + 1)\,dx$

$= \frac{9x^3}{3} - \frac{e^{2x}}{2} + x + C$

$= 3x^3 - \frac{1}{2}e^{2x} + x + C$

$\because F(1) = 4$,

$4 = 3(1)^3 - \frac{e^{2(1)}}{2} + (1) + C$

$4 = 3 - \frac{1}{2}e^2 + 1 + C$

$\frac{1}{2}e^2 = C$

$\therefore F(x) = 3x^3 - \frac{1}{2}e^{2x} + x + \frac{1}{2}e^2$

③ $f(x) = \int \left[\sin(2x) - \cos\left(\frac{x}{3}\right)\right] dx$

$= -\frac{1}{2}\cos(2x) - 3\sin\left(\frac{x}{3}\right) + C$

$\because f\left(\frac{\pi}{2}\right) = -3$

$-3 = -\frac{1}{2}\cos\left[2\left(\frac{\pi}{2}\right)\right] - 3\sin\left[\frac{\left(\frac{\pi}{2}\right)}{3}\right] + C$

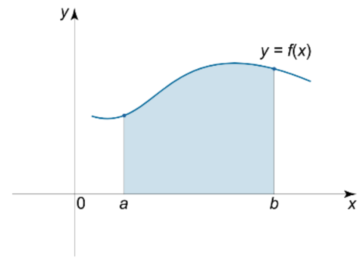
$-3 = -\frac{1}{2}\cos(\pi) - 3\sin\left(\frac{\pi}{6}\right) + C$

$-3 = -\frac{1}{2}(-1) - 3\left(\frac{1}{2}\right) + C$

$-2 = C$

$\therefore f(x) = \frac{-1}{2}\cos(2x) - 3\sin\left(\frac{x}{3}\right) - 2$

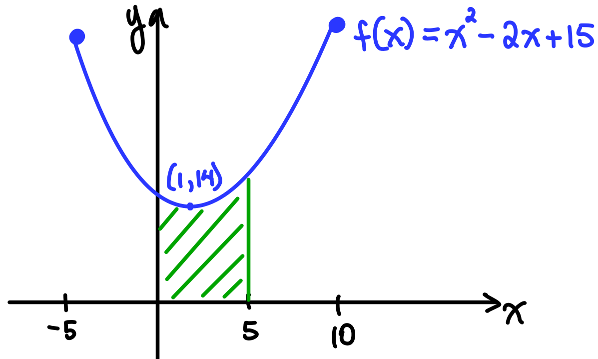
Area Under a Curve



The area enclosed between $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is written as $A = \int_a^b f(x)dx$ and can be evaluated. Area= F(b)-F(a)

Example 1: The function f is defined by $f(x) = x^2 - 2x + 15$.

a) Draw the graph of $y = f(x)$ for the domain $-5 \le x \le 10$.



b) Shade the area enclosed by the graph of $y = f(x)$, the x-axis, and the lines $x = 0$ and $x = 5$.

c) Evaluate the definite integral $\int_0^5 f(x)dx$ to find the area of the shaded region.

$$\text{Area} = \int_0^5 (x^2 - 2x + 15)\,dx$$

$$= \left.\frac{x^3}{3} - \frac{2x^2}{2} + 15x\right]_0^5$$

$$= \left.\frac{x^3}{3} - x^2 + 15x\right]_0^5$$

$$= \left[\frac{(5)^3}{3} - (5)^2 + 15(5)\right] - \left[\frac{(0)^3}{3} - (0)^2 + 15(0)\right]$$

$$= \frac{275}{3}$$

$$\doteq 91.7 \text{ units}^2$$

d) Check your answer to part (c) using your GDC.

GDC INSTRUCTIONS:

For TI-83:

Method 1: From the HOME SCREEN

1. Press MATH 9 fnInt(
2. Enter the function , x, T, θ , n , enter the value of the lower limit , upper limit) ENTER

Method 2: From the GRAPH

1. Enter the equation in the equation editor screen y= and graph the function.
2. Enter 2nd trace 7 , then enter the lower limit, press ENTER , enter the upper limit, then press ENTER .

For TI-84:

Method 1: From the HOME SCREEN

1. Press $\boxed{\text{MATH}}$ $\boxed{9}$ fnInt(.
2. The screen will display the general definite integral for you to enter the limits of integration values, the integrand and the variable you are integrating with respect to. Press $\boxed{\text{ENTER}}$ when you have entered all the information to evaluate.

Method 2: From the GRAPH

1. Enter the equation in the equation editor screen $\boxed{\text{y=}}$ and graph the function.
2. Enter $\boxed{2^{\text{nd}}}$ $\boxed{\text{trace}}$ $\boxed{7}$, then enter the lower limit, press $\boxed{\text{ENTER}}$, enter the upper limit, then press $\boxed{\text{ENTER}}$.

Example 2: Find the area of the region enclosed between the curve $y = 9 - x^2$ and the x-axis.

1) Find x-int for limits of integration:

$0 = 9 - x^2$

$x^2 = 9$

$x = \pm 3$

2) Find the area:

$$\text{Area} = \int_{-3}^{3} (9 - x^2)\,dx$$

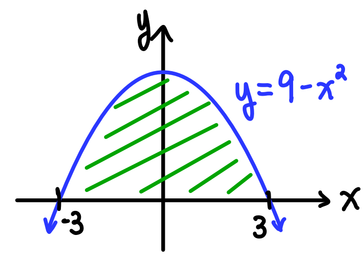
$$= 9x - \frac{x^3}{3}\Big]_{-3}^{3}$$

$$= \left[9(3) - \frac{(3)^3}{3}\right] - \left[9(-3) - \frac{(-3)^3}{3}\right]$$

$$= 18 - (-18)$$

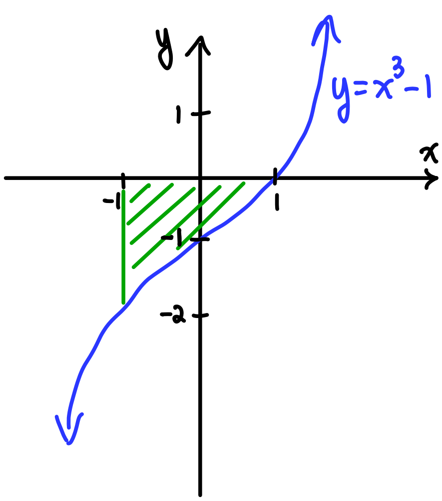
$$= 36$$

$\therefore$ the area is $36\,u^2$.



To determine the area enclosed between $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ where the function is **below the x-axis**, the definite integral becomes: $A = -\int_a^b f(x)dx$ or $A = \left|\int_a^b f(x)dx\right|$.

Example 3: Find the area between $y = x^3 - 1$ and the x-axis from $x = -1$ to $x = 1$.



$$\text{Area} = -\int_{-1}^{1} (x^3 - 1)\,dx$$

$$= -\left[\frac{x^4}{4} - x\right]_{-1}^{1}$$

$$= -\left[\left[\frac{(1)^4}{4} - (1)\right] - \left[\frac{(-1)^4}{4} - (-1)\right]\right]$$

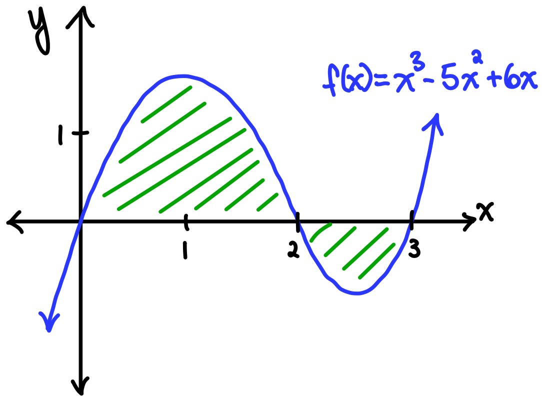
$$= -\left[-\frac{3}{4} - \frac{5}{4}\right]$$

$$= 2$$

$\therefore$ the area under the curve is 2 units^2.

Example 4: Consider the graph of the function $f(x) = x^3 - 5x^2 + 6x$.

a) Draw the graph of $y = f(x)$ over a suitable domain.



$$
\begin{aligned}
f(x) &= x^3 - 5x^2 + 6x \\
&= x(x^2 - 5x + 6) \\
&= x(x-3)(x+2)
\end{aligned}
$$

b) Determine the definite integral $\int_0^3 f(x)dx$.

$$
\begin{aligned}
\int_0^3 (x^3 - 5x^2 + 6x)\,dx &= \left.\frac{x^4}{4} - \frac{5x^3}{3} + \frac{6x^2}{2}\right]_0^3 \\
&= \left.\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_0^3
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{(3)^4}{4} - \frac{5(3)^3}{3} + 3(3)^2\right] - (0) \\
&= \frac{9}{4}
\end{aligned}
$$

c) Evaluate the definite integrals $\int_0^2 f(x)dx$ and $\int_2^3 f(x)dx$.

$$
\begin{aligned}
&\int_0^2 (x^3 - 5x^2 + 6x)\,dx \\
&= \left.\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_0^2 \\
&= \left[\frac{(2)^4}{4} - \frac{5(2)^3}{3} + 3(2)^2\right] - (0) \\
&= \frac{8}{3}
\end{aligned}
$$

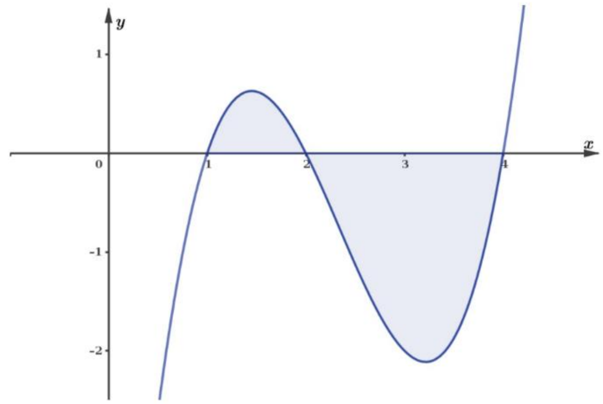
$$
\begin{aligned}
&\int_2^3 (x^3 - 5x^2 + 6x)\,dx \\
&= \left.\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_2^3 \\
&= \left[\frac{(3)^4}{4} - \frac{5(3)^3}{3} + 3(3)^2\right] - \left[\frac{(2)^4}{4} - \frac{5(2)^3}{3} + 3(2)^2\right] \\
&= \frac{-5}{12}
\end{aligned}
$$

d) Find the **area** enclosed between the graph of $y = f(x)$ and the x-axis.

$$
\begin{aligned}
\text{Area} &= \int_0^2 (x^3 - 5x^2 + 6x)\,dx - \int_2^3 (x^3 - 5x^2 + 6x)\,dx \\
&= \left.\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_0^2 - \left[\frac{x^4}{4} - \frac{5x^3}{3} + 3x^2\right]_2^3 \\
&= \frac{8}{3} - \left(-\frac{5}{12}\right) \\
&= \frac{37}{12}
\end{aligned}
$$

$\therefore$ The area is $\frac{37}{12}$ units^2.

The function of $f(x) = x^3 - 7x^2 + 14x - 8$ is shown to the right. There are x-intercepts of $x = 1, x = 2$ and $x = 4$.



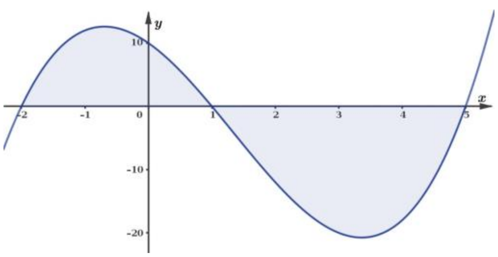
The definite integral, $A = \int_1^4 f(x)dx$ will give the **difference** between the two areas on the interval [1, 4].

If the **total area enclosed** between $f(x)$ and the x-axis is to be determined, we need to **split the integral** using the points of intersection between $f(x)$ and the x-axis (i.e. the x-intercepts). Then, we need to determine the definite integrals using the intervals [1, 2] and [2, 4] as the limits. Finally, adding the **magnitudes** of the definite integrals will determine the total area.

$$\therefore Area = \int_1^2 f(x)dx - \int_2^4 f(x)dx \quad \text{or} \quad Area = \int_1^2 f(x)dx + \left|\int_2^4 f(x)dx\right|.$$

Note: Displacement $= \int_a^b v(t)\,dt$ but distance $= \int_a^b |v(t)|\,dt$ (refer to CP 12)

Note:



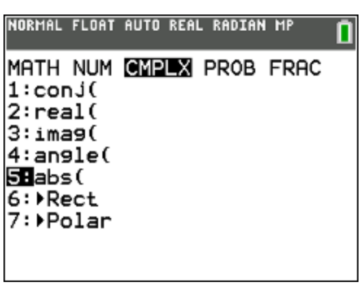
If the area of the region bounded by the graph of $y = f(x)$ and the x-axis is calculated by hand, then the parts **above and below the x-axis** need to be considered **separately**.

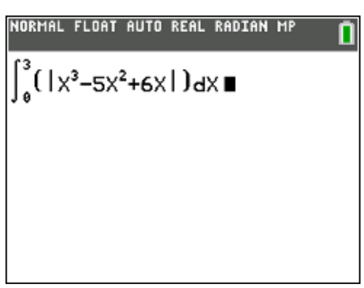
If the area is calculated by the **GDC**, then the formula $\int_a^b |f(x)|dx$ can be used. The similar formula without the absolute value only gives the correct answer if the entire graph of f is above the x-axis.

GDC INSTRUCTIONS

Note: To use the GDC to enter $\int_a^b |f(x)|dx$, the absolute value function needs to be entered. In the integrand, press MATH , arrow to the **right** to the CMPLX menu, select 5 for 5: abs(, then enter the function inside the absolute value sign.

Refer to pages 19-20 for full instructions on getting to the Integration option.





1-5 Warm Up

1. Determine the anti-derivative, $F(x)$:

a) $f(x) = 12x^3 - 9x^2 + 8x + 31$

b) $f(x) = 4\sin(2x) + 5\cos(3x+1)$

c) $f(x) = -2e^{\sqrt{3x}} + \frac{\sqrt{2}}{x+1}$

d) $f(x) = x^2\sqrt{x^3+2}$

2. Determine the integrals:

a) $\int e^{\cos(x)}\sin(x)\,dx$

b) $\int \frac{\sqrt{\ln(x)}}{x}\,dx$

c) $\int \sqrt{1+\tan(x)}\,\sec^2(x)\,dx$

d) $\int \cos^3(x)\,dx$

3. A particle is accelerated in a line so that its velocity in meters per second is equal to the square root of the time elapsed, in seconds. How far does the particle travel in the first hour?

Solutions:

① a) $F(x) = \frac{12x^4}{4} - \frac{9x^3}{3} + \frac{8x^2}{2} + 31x + c$

$= 3x^4 - 3x^3 + 4x^2 + 31x + c$

b) $F(x) = \frac{4[-\cos(2x)]}{2} + \frac{5\sin(3x+1)}{3} + c$

$= -2\cos(2x) + \frac{5}{3}\sin(3x+1) + c$

c) $F(x) = \frac{-2e^{3x}}{3} + \sqrt{2}\ln|x+1| + c$

$= -\frac{2}{3}e^{3x} + \sqrt{2}\ln|x+1| + c$

d) $F(x) = \frac{x^2(x^3+2)^{\frac{3}{2}}}{\frac{3}{2}(3x^2)} + c$

$= \frac{2}{9}(x^3+2)^{\frac{3}{2}} + c$

② a) $\int e^{\cos x}\sin x\,dx$

$= \frac{e^{\cos x}\sin x}{-\sin x} + c$

$= -e^{\cos x} + c$

b) $\int \frac{\sqrt{\ln(x)}}{x}\,dx$

$= \frac{[\ln(x)]^{\frac{3}{2}}}{\frac{3}{2}\left(\frac{1}{\cancel{x}}\right)\cancel{x}} + c$

$= \frac{2}{3}[\ln(x)]^{\frac{3}{2}} + c$

Solutions continued...

c) $\int \sqrt{1+\tan(x)}\, \sec^2(x)\, dx$

Let $u = 1+\tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\, dx$

$\therefore \int u^{\frac{1}{2}}\, du$

$= \frac{2}{3} u^{\frac{3}{2}} + C$

$= \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + C$

d) $\int [\cos^3(x)]\, dx$

$= \int \cos^2(x)\cos(x)\, dx$

$= \int [1-\sin^2(x)]\cos(x)\, dx$

Let $u = \sin(x)$

$\frac{du}{dx} = \cos(x)$

$du = \cos(x)\, dx$

$\therefore = \int (1-u^2)\, du$

$= u - \frac{1}{3}u^3 + C$

$\therefore = \sin(x) - \frac{1}{3}\sin^3(x) + C$

③ A particle is accelerated in a line so that its velocity in meters per second is equal to the square root of the time elapsed, in seconds. How far does the particle travel in the 1st hour?

$\frac{ds}{dt} = \sqrt{t}$

$s = \int t^{\frac{1}{2}}\, dt$

$= \frac{2}{3}t^{\frac{3}{2}} + C$

$\because t=0 \rightarrow s=0$

$\therefore C = 0$

$\therefore s = \frac{2}{3}t^{\frac{3}{2}}$

@ $t = 3600$, $s = \frac{2}{3}(3600)^{\frac{3}{2}}$

$= \frac{2}{3}(60)^3$

$= 144\,000$

$\therefore$ The particle travels 144 km in the first hour.

Alternate solution: $s = \int_0^{3600} t^{\frac{1}{2}}\, dt$

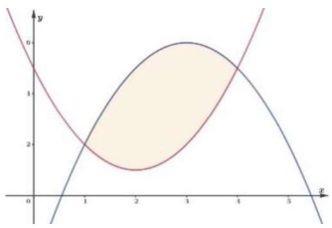
Area Between Two Curves

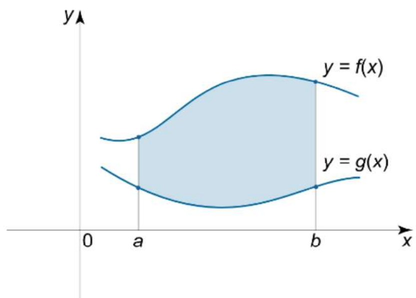
In general, if f and g are continuous functions in [a, b] and $f \geq g$ in $[a, b]$, then the area between f and g from a to b is the area under $f - g$ from a to b.

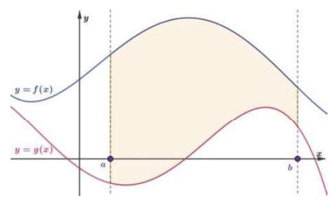
$$A = \int_a^b (f(x) - g(x))dx, \qquad f(x) \geq g(x)$$

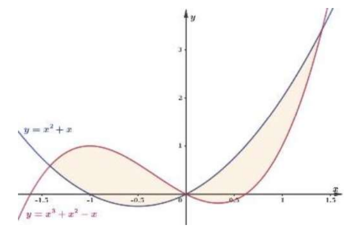
Procedure:

- Sketch the graph, if possible.
- Find points of intersection of curves to decide how many areas need to be calculated

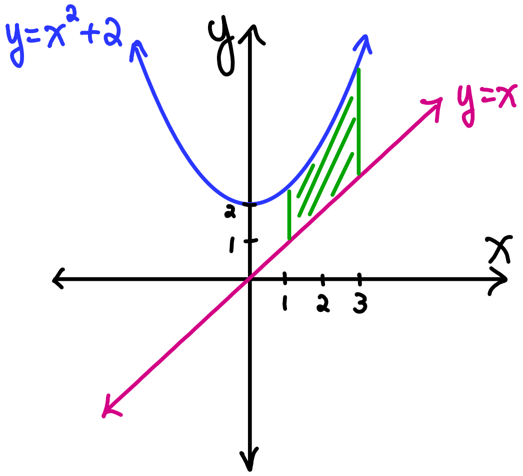








Example 1: Find the area between $y = x^2 + 2$ and $y = x$ from $x = 1$ to $x = 3$.



$$\text{Area} = \int_1^3 [(x^2+2)-(x)]dx$$

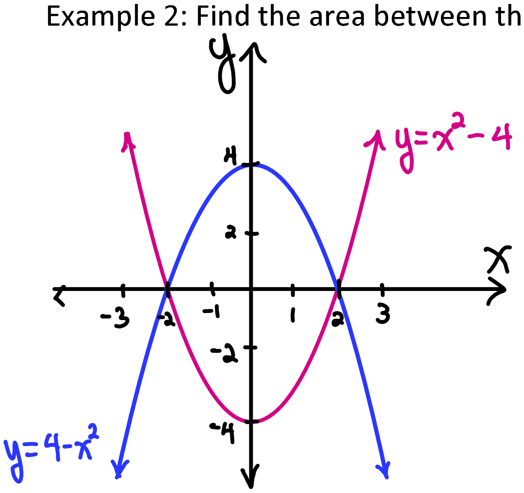
$$= \frac{x^3}{3} + 2x - \frac{x^2}{2}\Big]_1^3$$

$$= \left[\frac{(3)^3}{3} + 2(3) - \frac{(3)^2}{2}\right] - \left[\frac{(1)^3}{3} + 2(1) - \frac{(1)^2}{2}\right]$$

$$= \frac{26}{3}$$

$\therefore$ the area is $\frac{26}{3}$ units^2.

Example 2: Find the area between the graphs $y = 4 - x^2$ and $y = x^2 - 4$.



1) Find the pt. of int. for limits:

$$4 - x^2 = x^2 - 4$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$x = \pm 2$$

2) Find the area:

$$A = \int_{-2}^{2} [(4-x^2)-(x^2-4)]dx$$

$$= \int_{-2}^{2} (8-2x^2)dx$$

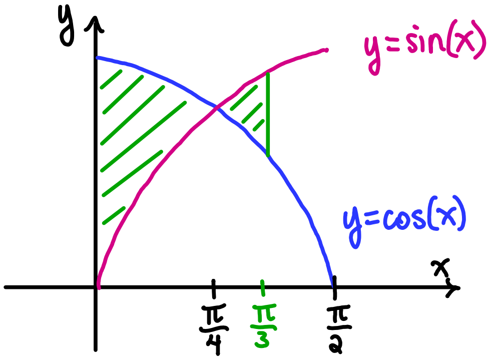
$$= 8x - \frac{2x^3}{3}\Big]_{-2}^{2}$$

$$= \left[8(2) - \frac{2(2)^3}{3}\right] - \left[8(-2) - \frac{2(-2)^3}{3}\right]$$

$$= \frac{64}{3}$$

$\therefore$ the area is $\frac{64}{3}$ u^2.

Example 3: Find the area between the graphs $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \frac{\pi}{3}$.



1) Intersection point(s):

$\sin(x) = \cos(x)$

$\tan(x) = 1$

$x = \frac{\pi}{4}$

2) Find area:

$$\text{Area} = \int_0^{\frac{\pi}{4}} [\cos(x) - \sin(x)]\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} [\sin(x) - \cos(x)]\,dx$$

$$= \Big[\sin(x) + \cos(x)\Big]_0^{\frac{\pi}{4}} + \Big[-\cos(x) - \sin(x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left[\left(\sin\left(\tfrac{\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) - \left(\sin(0) - \cos(0)\right)\right] +$$

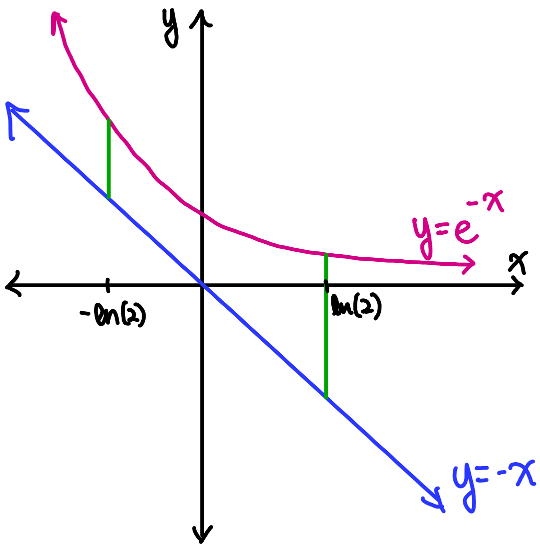
$$\left[\left(-\cos\left(\tfrac{\pi}{3}\right) - \sin\left(\tfrac{\pi}{3}\right)\right) - \left(-\cos\left(\tfrac{\pi}{4}\right) - \sin\left(\tfrac{\pi}{4}\right)\right)\right]$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$\doteq 0.462$$

$\therefore$ the area is approx. 0.462 u^2.

Example 4: Find the area between the graphs $y = e^{-x}$ and $y = -x$ from $x = -\ln 2$ to $x = \ln 2$.



$$\text{Area} = \int_{-\ln(2)}^{\ln(2)} \left[e^{-x} - (-x)\right] dx$$

$$= -e^{-x} + \frac{x^2}{2}\Big]_{-\ln(2)}^{\ln(2)}$$

$$= \left[-e^{-\ln(2)} + \frac{[\ln(2)]^2}{2}\right] - \left[-e^{-(-\ln(2))} + \frac{[-\ln(2)]^2}{2}\right]$$

$$= \left[-e^{\ln(2)^{-1}} + \frac{[\ln(2)]^2}{2}\right] - \left[-e^{\ln(2)} + \frac{[\ln(2)]^2}{2}\right]$$

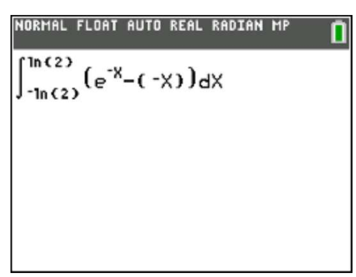
$$= \left[-\frac{1}{2} + \frac{[\ln(2)]^2}{2}\right] - \left[-2 + \frac{[\ln(2)]^2}{2}\right]$$

$$= \frac{3}{2}$$

$\therefore$ the area is $\frac{3}{2}$ units^2.

GDC INSTRUCTIONS

Note: To use the GDC to evaluate the definite integral, enter $f(x) - g(x)$ in the integrand. Refer to pages 19-20 for full instructions.



The Area between Two Curves

So far, we have only considered areas bounded by a single curve and the x-axis. Sometimes it is useful to be able to find areas bounded by two curves.

There are a few scenarios that you will have to consider when answering a question

1. **Finding the area of a region between two curves**

Ex) Find the area of the region bounded by the graphs of $y = x^2 + 2, y = -x, x = 0, and\ x = 1.$ $\left[\frac{17}{6}\right]$

2. **Area of a region between intersecting curves**

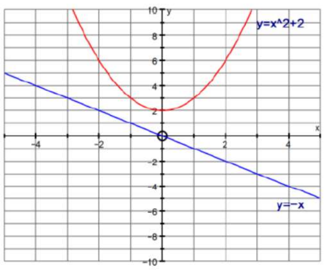
Ex) Find the area of the region bounded by the graphs of $f(x) = 2 - x^2\ and\ g(x) = x.$ $\left[\frac{9}{2}\right]$

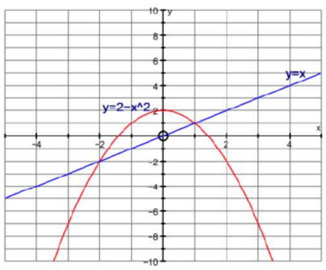
3. **Region lying between two intersecting curves**

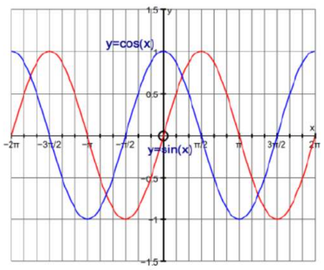
Ex) The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of one of these regions. $[2\sqrt{2}]$

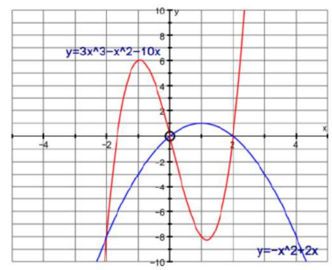
4. **Curves that intersect at more than two points**

Ex) Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$. $[24]$









<u>**Area of a Region between Two Curves**</u>

If f and g are continuous on $[a, b]$ and $f(x) \geq g(x)$ for all $x \epsilon [a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

In order to do these problems, you must

1. Find the intersection points of $f(x)\ and\ g(x)$.
2. Determine when each curve is above the other in the specified interval $[a, b]$.
3. Write down separate integrals to calculate the area where appropriate.
4. Calculate each integral and find the total area.

For difficult **polynomial** problems (case 4), the best tool for completing steps 1-3 is by solving $f(x) \geq g(x)$ using an interval table

- Ie) Create an interval table to solve $[f(x) - g(x)] \geq 0$. In doing this, you will find the x-values where f(x) and g(x) intersect. When you get +, this means f(x)>g(x) and when you get -, this means f(x)<g(x). This allows you to set up your integrals properly.

Sample Problems

1. Find the area enclosed between the graphs of $y = 2 - x$ and $y = 12 - (x - 2)^2$ $\left[\frac{343}{6}\right]$

2. Find the area enclosed by the graphs of $y = e^x$ and $y = x^2$, the y-axis, and the line $x = 2$. $\left[e^2 - \frac{11}{3}\right]$

3. Find the area enclosed between the graphs of $y = 2\cos(x) + 1$ and $y = 1 - 2\sin(x)$ where $0 \le x \le \pi$. $[4\sqrt{2}]$
   - Hint: you will have to solve $\tan(x) = -1$

4. Find the area enclosed between the graphs of $y = x^3 + x$ and $y = 2x$ where $-2 \le x \le 1$. $\left[\frac{11}{4}\right]$

5. Find the total area enclosed between the graphs of $y = x(x - 4)^2$ and $y = x^2 - 7x + 15$. $[8]$

6. The area enclosed between the curve $y = x^2$ and the line $y = mx$ is $\frac{32}{3}$. Find the value of m if $m > 0$. $[m = 4]$

Area Between Two Curves Sample Problems-Solutions

1. $y_1 = 2-x \qquad y_2 = 12-(x-2)^2$ testpt: $x=3$

$$2-x = 12-(x-2)^2$$

$$2-x = 12-(2-x)^2$$

Let $a = 2-x$

$$a = 12-a^2$$

$$a^2+a-12=0$$

$$(a+4)(a-3)=0$$

$$a=-4 \text{ or } a=3$$

$$2-x=-4 \qquad 2-x=3$$

$$x=6 \qquad x=-1$$

$y_1 = -1$

$y_2 = 12-(1)^2$

$= 11$ ✓

$$\text{Area} = \int_{-1}^{6} \left[(12-(x-2)^2) - (2-x)\right] dx$$

$$= \int_{-1}^{6} \left[(12-x^2+4x-4) -2+x\right] dx$$

$$= \int_{-1}^{6} (-x^2+5x+6)\, dx$$

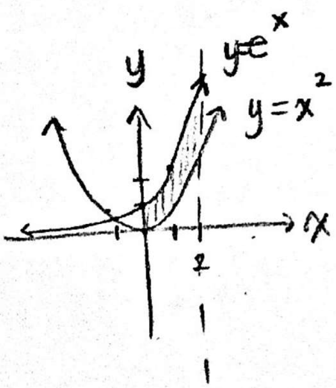
$$= \left[-\frac{x^3}{3}+\frac{5}{2}x^2+6x\right]_{-1}^{6}$$

$$= \left[-72+90+36\right]-\left[\frac{1}{3}+\frac{5}{2}-6\right]$$

$$= 54+\frac{19}{6}$$

$$= \frac{343}{6}$$

2.



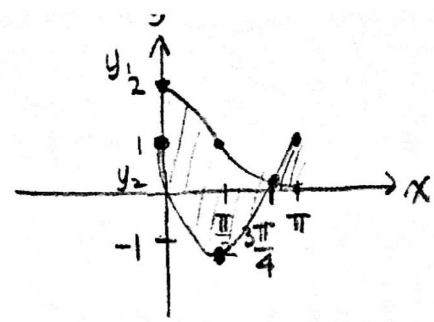
$$\text{Area} = \int_0^2 (e^x - x^2)\, dx$$

$$= \left[e^x - \frac{x^3}{3}\right]_0^2$$

$$= \left(e^2-\frac{8}{3}\right)-1$$

$$= e^2-\frac{11}{3}$$

3. $y_1 = 2\cos x + 1 \qquad y_2 = 1 - 2\sin x$



$$2\cos x + \cancel{1} = \cancel{1} - 2\sin x$$

$$\tan x = -1$$

ref $x = \tan^{-1}(1)$

$= \frac{\pi}{4}$

$x_1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \qquad x_2 = $ X outside domain

$$\text{Area} = \int_0^{\frac{3\pi}{4}} [(2\cos x + 1) - (1 - 2\sin x)]\,dx + \int_{\frac{3\pi}{4}}^{\pi} [(1 - 2\sin x) - (2\cos x + 1)]\,dx$$

$$= \int_0^{\frac{3\pi}{4}} (2\cos x + 2\sin x)\,dx + \int_{\frac{3\pi}{4}}^{\pi} (-2\sin x - 2\cos x)\,dx$$

$$= \Big[2\sin x - 2\cos x\Big]_0^{\frac{3\pi}{4}} + \Big[+2\cos x - 2\sin x\Big]_{\frac{3\pi}{4}}^{\pi}$$

$$= \left(\frac{2}{\sqrt2} + \frac{2}{\sqrt2}\right) - (-2) + \left[(-2) - \left(-\frac{2}{\sqrt2} - \frac{2}{\sqrt2}\right)\right]$$

$$= \frac{4}{\sqrt2} + 2 + \left[-2 + \frac{4}{\sqrt2}\right]$$

$$= \frac{8}{\sqrt2} = 4\sqrt2.$$

4. $y_1 = x^3 + x \qquad y_2 = 2x$

$$x^3 + x = 2x$$

$$x^3 - x = 0$$

$$x(x^2 - 1) = 0$$

$$x = 0, \pm 1$$

	$(-2,-1)$	$(-1,0)$	$(0,1)$
x	−	−	+
$x-1$	−	−	−
$x+1$	−	+	+
$y_1 - y_2$	−	+	−
	y_2	y_1	y_2

$$\text{Area} = \int_{-2}^{-1} (2x - x^3 - x)\,dx + \int_{-1}^{0} (x^3 + x - 2x)\,dx + \int_0^1 (2x - x^3 - x)\,dx$$

$$= \left[x^2 - \frac{x^4}{4} - \frac{x^2}{2}\right]_{-2}^{-1} + \left[\frac{x^4}{4} + \frac{x^2}{2} - x^2\right]_{-1}^{0} + \left[x^2 - \frac{x^4}{4} - \frac{x^2}{2}\right]_0^1$$

$$= [(0.25) - (-2)] + (+0.25) + 0.25 = 2.75$$

5. $y_1 = x(x-4)^2 \qquad y_2 = x^2 - 7x + 15$

Int. pts: $x(x^2-8x+16) = x^2-7x+15$

$x^3-8x^2+16x = x^2-7x+15$

$x^3-9x^2+23x-15=0$

$\therefore\ x-1$ is a factor

$$
\begin{array}{r}
x^2 - 8x + 15 \\
x-1\,\overline{)\,x^3-9x^2+23x-15} \\
\underline{x^3 - x^2} \qquad\qquad \\
-8x^2+23x \qquad \\
\underline{-8x^2+8x} \qquad \\
15x-15 \\
\underline{15x-15} \\
0
\end{array}
$$

$\therefore\ x^3-9x^2+23x-15 = (x-1)(x^2-8x+15)$

$= (x-1)(x-3)(x-5)$

$x = 1, 3, 5$

	$(1,3)$	$(3,5)$
$x-1$	+	+
$x-3$	−	+
$x-5$	−	−
y_1-y_2	+	−
	y_1	y_2

Area $= \int_1^3 (x^3-9x^2+23x-15)\,dx + \int_3^5 (-x^3+9x^2-23x+15)\,dx$

$= \left[\frac{x^4}{4} - 3x^3 + \frac{23}{2}x^2 - 15x\right]_1^3 + \left[-\frac{x^4}{4} + 3x^3 - \frac{23}{2}x^2 + 15x\right]_3^5$

$= 4+4$

$= 8$

6. $y_1 = x^2 \quad y_2 = mx$

$x^2 = mx$

$x^2 - mx = 0$

$x(x-m) = 0$

$x = 0, m$

Area $= \int_0^m [mx - x^2]\,dx$

$\frac{32}{3} = \left[\frac{mx^2}{2} - \frac{x^3}{3}\right]_0^m$

$\frac{32}{3} = \frac{m^3}{2} - \frac{m^3}{3}$

$64 = m^3$

$m = 4$

Note: if you chose x^2 to be above mx, area would be neg.

Warm Up

1. Determine the area of the region bounded by $y = -x$ and $y = x^2 + 2$, $x = 0$ and $x = 1$.

$$A = \int_0^1 (x^2 + 2 + x)\,dx$$

$$= \frac{x^3}{3} + 2x + \frac{x^2}{2} \Big]_0^1$$

$$= \left[\frac{(1)^3}{3} + 2(1) + \frac{(1)^2}{2}\right] - 0$$

$$= \frac{17}{6} \quad \therefore \text{ The area is } \frac{17}{6}u^2$$

2. Determine the area between the curves $f(x) = x - 1$ and $g(x) = x^2 + 3x - 4$ from $x = -5$ to $x = 2$.

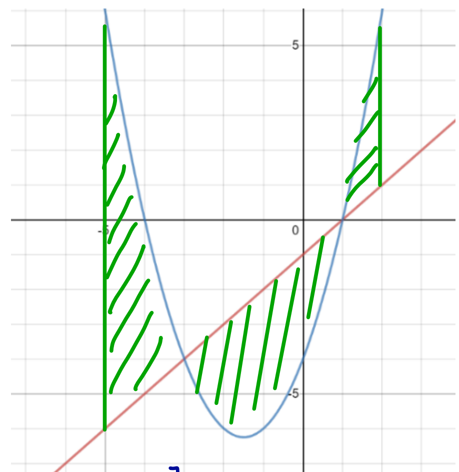
Point(s) of Intersection:

$$x^2 + 3x - 4 = x - 1$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$$\therefore x = -3,\ x = 1$$



$$A = \int_{-5}^{-3} [(x^2+3x-4)-(x-1)]\,dx + \int_{-3}^{1} [(x-1)-(x^2+3x-4)]\,dx + \int_{1}^{2} [(x^2+3x-4)-(x-1)]\,dx$$

$$= \int_{-5}^{-3} (x^2-2x-3)\,dx - \int_{-3}^{1} (x^2+2x-3)\,dx + \int_{1}^{2} (x^2+2x-3)\,dx$$

$$= \frac{x^3}{3} + x^2 - 3x\Big]_{-5}^{-3} - \left[\frac{x^3}{3} + x^2 - 3x\right]_{-3}^{1} + \left[\frac{x^3}{3} + x^2 - 3x\right]_{1}^{2}$$

$$= \left[\frac{(-3)^3}{3} + (-3)^2 - 3(-3)\right] - \left[\frac{(-5)^3}{3} + (-5)^2 - 3(-5)\right] - \left[\frac{(1)^3}{3} + (1)^2 - 3(1)\right] + \left[\frac{(-3)^3}{3} + (-3)^2 - 3(-3)\right]$$

$$+ \left[\frac{(2)^3}{3} + (2)^2 - 3(2)\right] - \left[\frac{(1)^3}{3} + (1)^2 - 3(1)\right]$$

$$= 9 - \left(-\frac{5}{3}\right) - \left(-\frac{5}{3}\right) + 9 + \frac{2}{3} - \left(\frac{-5}{3}\right)$$

$$= \frac{32}{3} + \frac{32}{3} + \frac{7}{3}$$

$$= \frac{71}{3}u^2$$

1-6 Warm Up

1. Determine the integrals:

   a) $\int \frac{2+x}{x^3} dx$

   b) $\int \frac{dx}{x+5}$

   c) $\int \frac{3x}{(x^2+3)^5} dx$

   d) $\int \sin(x)\,(1-\sin^2(x))\,dx$

2. Evaluate:

   a) $\int_2^5 (2x^3 - 3x^2 + 7x + 2)dx$

   b) $\int_1^8 \frac{1}{\sqrt[3]{x^2}} dx$

3. Determine the area below the curve $y = x^2 + 6$ between $x = 1$ and $x = 4$.

4. Determine the area enclosed between the curve $y = x^3 - 3x^2 - 10x$ and the x-axis. Verify your answer using the GDC.

Solutions:

① a) $\int (2x^{-3} + x^{-2})dx$

$= -\frac{1}{2}(2x^{-2}) - x^{-1} + c$

$= -\frac{1}{x^2} - \frac{1}{x} + c$

b) $\ln|x+5| + c$

c) let $u = x^2+3$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$

$\therefore \int \frac{3}{2u^5} du$

$= \frac{3}{2}\left(\frac{u^{-4}}{-4}\right) + c$

$= -\frac{3}{8(x^2+3)^4} + c$

d) $\int \sin x(\cos^2 x)\,dx$

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x\,dx$

$\therefore -\int u^2 du$

$= -\frac{u^3}{3} + c$

$= -\frac{\cos^3 x}{3} + c$

1-6 Warm Up solutions continued

② a) $\frac{2x^4}{4} - \frac{3x^3}{3} + \frac{7x^2}{2} + 2x \Big]_2^5$

$= \frac{1}{2}x^4 - x^3 + \frac{7}{2}x^2 + 2x \Big]_2^5$

$= \left[\frac{1}{2}(5)^4 - (5)^3 + \frac{7}{2}(5)^2 + 2(5)\right] - \frac{1}{2}\left[(2)^4 - (2)^3 + \frac{7}{2}(2)^2 + 2(2)\right]$

$= 285 - 18$

$= 267$

b) $\int_1^8 x^{-\frac{2}{3}}\,dx$

$= 3x^{\frac{1}{3}} \Big]_1^8$

$= 3(8)^{\frac{1}{3}} - 3(1)^{\frac{1}{3}}$

$= 6 - 3$

$= 3$

③ Area $= \int_1^4 (x^2+6)\,dx$

$= \frac{1}{3}x^3 + 6x \Big]_1^4$

$= \left[\frac{1}{3}(4)^3 + 6(4)\right] - \left[\frac{1}{3}(1)^3 + 6(1)\right]$

$= \frac{64}{3} + 24 - \frac{1}{3} - 6$

$= 39$ $\therefore$ Area is 39 units^2

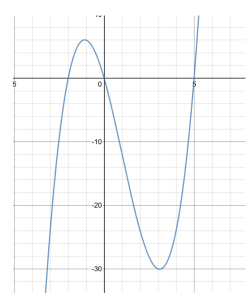
④ x-int:

$0 = x^3 - 3x^2 - 10x$

$0 = x(x-5)(x+2)$

$x = 0, 5, -2$

Area $= \int_{-2}^{0} (x^3 - 3x^2 - 10x)\,dx - \int_0^5 (x^3 - 3x^2 - 10x)\,dx$



$= \left[\frac{x^4}{4} - x^3 - 5x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - x^3 - 5x^2\right]_0^5$

$= 0 - \left[\frac{(-2)^4}{4} - (-2)^3 - 5(-2)^2\right] - \left[\frac{(5)^4}{4} - (5)^3 - 5(5)^2\right] + 0$

$= 8 - \left(-\frac{375}{4}\right)$

$= \frac{407}{4}$ $\therefore$ The area is $\frac{407}{4}$ or 101.75 units^2

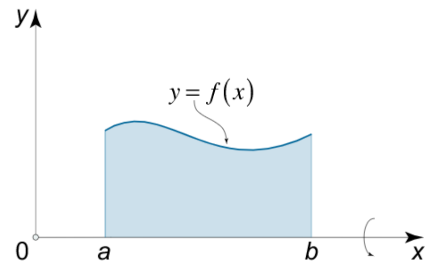
using GDC: $A = \int_{-2}^{5} |x^3 - 3x^2 - 10x|\,dx$

$= 101.75$

MATH, 9, enter upper and lower limits, go to the integrand, MATH, $\boxed{\blacktriangleright}$, Num menu, $\boxed{1}$: abs, then enter the function

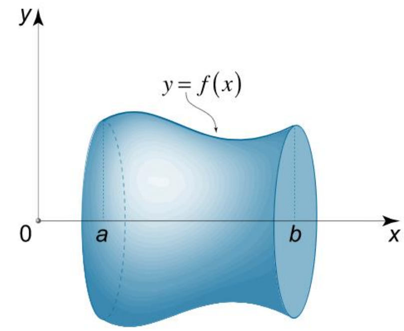
Volumes of Solids of Revolution

A cylinder, a cone and a sphere are all examples of **solids of revolution**, which are shapes formed by completely rotating a line or a curve about a fixed axis. The volumes of these shapes are called **volumes of solids of revolution**.

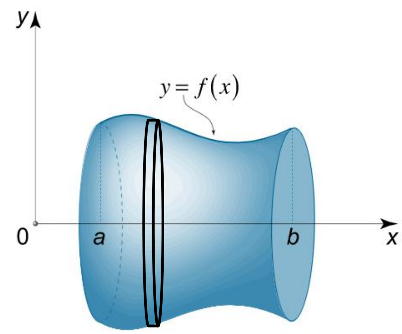


To get a solid of revolution, we start out with a function, $y = f(x)$, on an interval $[a, b]$.

We then **rotate** this curve about a given axis to get the surface of the solid of revolution. For purposes of simplicity, we will rotate the curve about the x-axis, although it could be any vertical or horizontal axis. Doing this to the curve will give the following three-dimensional region.



To find the volume of the solid of revolution, we can slice this solid into many disks from $[a, b]$. Each disk will be cylindrical in shape where the volume can be determined by the general formula, $V = \pi r^2 h$. Each disk in the solid will have a radius of y and thickness of dx.



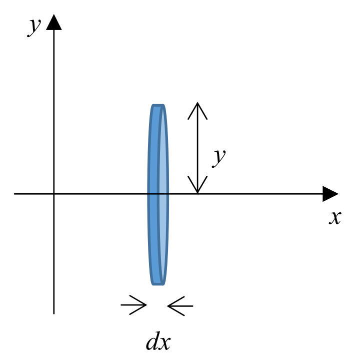
The volume, V, of the disk is given by: $V = \pi y^2 dx$

Volume, V, of the whole solid is the sum of all such disks from $x = a$ to $x = b$.

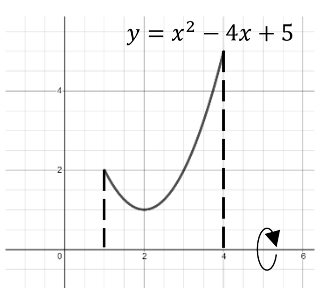
$$V = \sum_{x=a}^{b} \pi y^2 dx$$

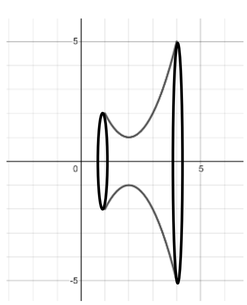
The smaller the disks the solid is divided into, the closer the value is to the actual volume. In other words, we want $x \to 0$.

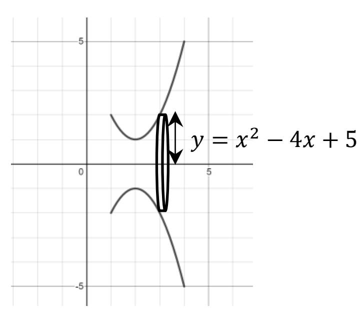
$$V = \lim_{x \to 0} \pi \sum_{x=a}^{b} y^2 dx \qquad \text{or} \qquad \boxed{V = \pi \int_a^b y^2 dx}$$



For example, determine the volume of the solid obtained by rotating the region bounded by $y = x^2 - 4x + 5$, $x = 1$, $x = 4$ and the x-axis about the x-axis for 2π radians.







In this case, the radius is the distance from the x-axis to the curve, which is the same as the function value at that particular x. The cross-sectional area is then,

$$\begin{aligned} A(x) &= \pi(x^2 - 4x + 5)^2 \\ &= \pi(x^4 - 8x^3 + 26x^2 - 40x + 25) \end{aligned}$$

Next, determine the limits of integration. From left to right, the first cross section will occur at $x = 1$ and the last cross section will occur at $x = 4$. These are the limits of integration.

The volume of the solid is then,

$$\begin{aligned} V &= \pi \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25)dx \\ &= \pi\left(\frac{1}{5}x^5 - 2x^4 + \frac{26}{3}x^3 - 20x^2 + 25x\right)\Big|_1^4 \\ &= \frac{78\pi}{5} \end{aligned}$$

Example 1. Find the volume of the solid formed when the area between the curve $y = x^2 + 1$ and the x-axis from $x = 1$ to $x = 3$ is rotated through 2π radians about the x-axis.

$$V = \pi \int_1^3 (x^2+1)^2 dx$$

$$= \pi \int_1^3 (x^4 + 2x^2 + 1)\, dx$$

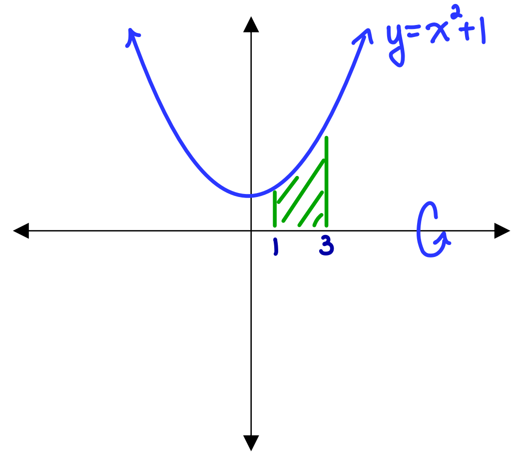
$$= \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_1^3 \quad \text{or} \quad \pi\left(\frac{x^5}{5} + \frac{2x^3}{3} + x\right|_1^3$$

$$= \pi\left[\left(\frac{(3)^5}{5} + \frac{2(3)^3}{3} + (3)\right) - \left(\frac{(1)^5}{5} + \frac{2(1)^3}{3} + (1)\right)\right]$$

$$= \pi\left(\frac{348}{5} - \frac{28}{15}\right)$$

$$= \frac{1016\pi}{15}$$

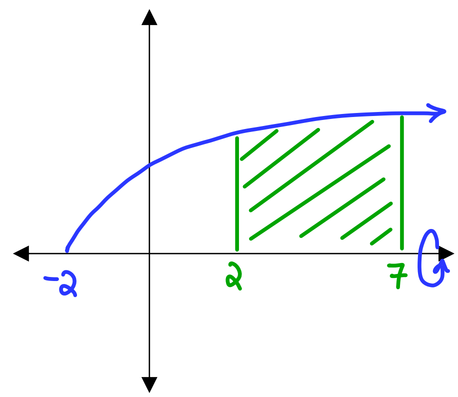
$\therefore$ The volume is $\frac{1016\pi}{15}$ units^3.



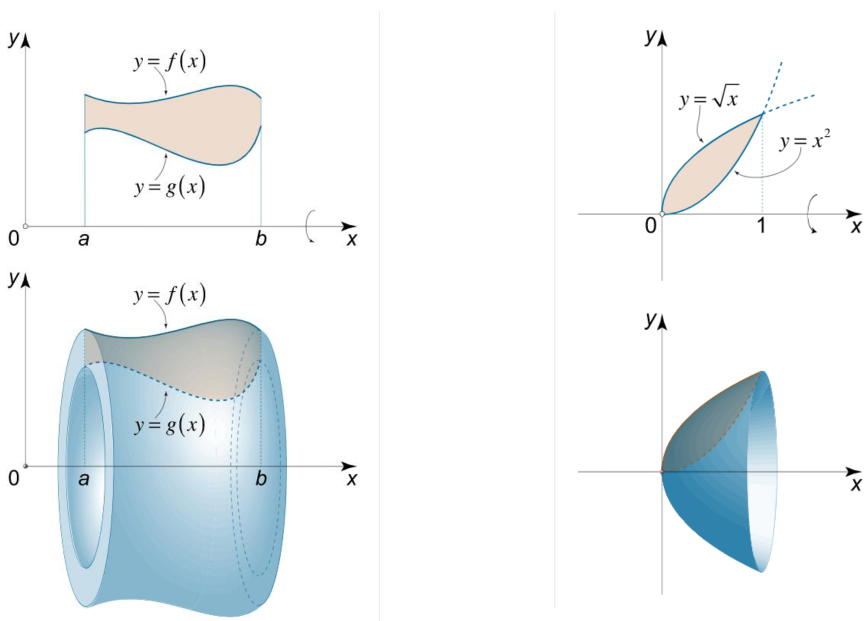
Example 2. Find the volume of the solid formed when the region enclosed by the curve $y=\sqrt{x+2}$ is completely rotated about the x-axis between $x=2$ and $x=7$.

$$V=\pi\int_2^7\left(\sqrt{x+2}\right)^2dx$$
$$=\pi\int_2^7(x+2)\,dx$$
$$=\pi\left[\frac{x^2}{2}+2x\right]_2^7$$
$$=\pi\left[\left(\frac{(7)^2}{2}+2(7)\right)-\left(\frac{(2)^2}{2}+2(2)\right)\right]$$
$$=\pi\left(\frac{49}{2}+14-2-4\right)$$
$$=\frac{65\pi}{2}$$

$\therefore$ the volume is $\frac{65\pi}{2}$ units^3.



You can also rotate the **area between 2 curves** about the x-axis to form a solid.



In this case, the formula for the volume of the solid of revolution becomes,

$$\boxed{V=\pi\int_a^b\left[(g(x))^2-(f(x))^2\right]dx\text{ , where }g(x)>f(x)}$$

Example 3. The area enclosed between the curve $y = 4 - x^2$ and the line $y = 4 - 2x$ is rotated about the x-axis. Find the volume of the solid generated.

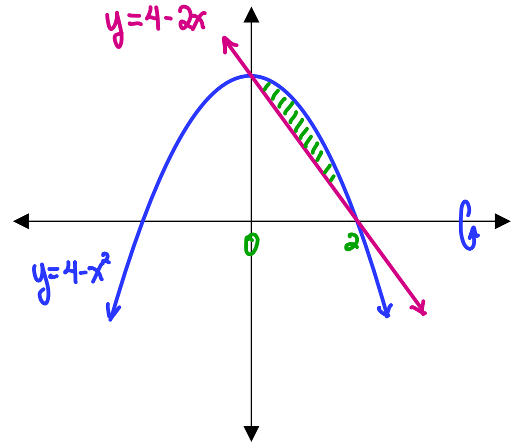
1) Point(s) of Intersection:

$4 - x^2 = 4 - 2x$

$x^2 - 2x = 0$

$x(x-2) = 0$

$\therefore x = 0$ or $x = 2$



2) Volume:

$V = \pi \int_0^2 \left[(4-x^2)^2 - (4-2x)^2\right] dx$

$= \pi \int_0^2 (16 - 8x^2 + x^4 - 16 + 16x - 4x^2)\, dx$

$= \pi \int_0^2 (x^4 - 12x^2 + 16x)\, dx$

$= \pi \left[\frac{x^5}{5} - 4x^3 + 8x^2\right]_0^2$

$= \pi \left[\left(\frac{(2)^5}{5} - 4(2)^3 + 8(2)^2\right) - (0)\right]$

$= \pi \left(\frac{32}{5} - 32 + 32\right)$

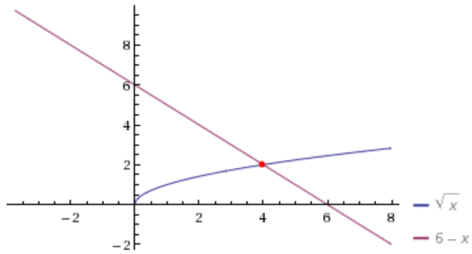
$= \frac{32\pi}{5}$

$\therefore$ the volume is $\frac{32\pi}{5}$ units^3.

Volume of Revolution about x-axis Problems

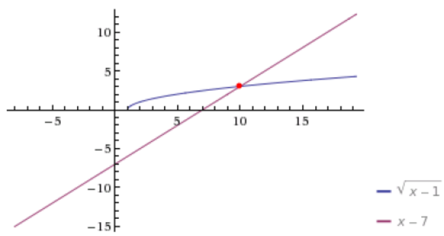
1. Find the volume of the solid of revolution generated when the area described is rotated about the x-axis
   a. The area between the curve $y = x^{\frac{3}{2}}$ and the ordinates $x = 1$ and $x = 3$. $[20\pi]$
   b. The area between the curve $x^2 + y^2 = 16$ and the ordinates $x = -1$ and $x = 1$. $\left[\frac{94}{3}\pi\right]$
   c. The area between the curve $y = (2 + x)^2$ and the ordinates $x = 0$ and $x = 1$. $\left[\frac{211}{5}\pi\right]$

2. Consider the region enclosed by the curves $y = \sqrt{x}$, $y = 6 - x$, and the x-axis. Rotate this region about the x-axis and find the resulting volume. $\left[\frac{32\pi}{3}\right]$



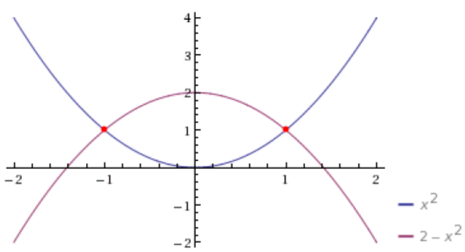
$$V = \pi \int_0^4 \left[\sqrt{x}\right]^2 dx + \pi \int_4^6 [6 - x]^2 dx$$

3. Let R be the region in the first quadrant enclosed by $y = \sqrt{x - 1}$, $y = x - 7$ and the x-axis. Sketch the region. Rotate R about the x-axis and find the resulting volume. $\left[\frac{63}{2}\pi\right]$



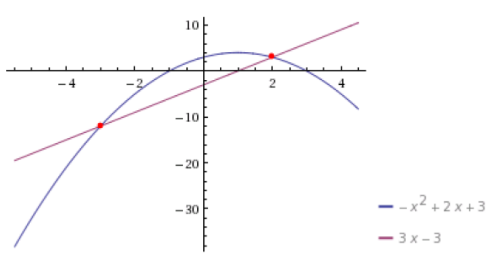
$$V = \pi \int_1^7 \left[\sqrt{x-1}\right]^2 dx + \pi \int_7^{10} \left[\left[\sqrt{x-1}\right]^2 - [x - 7]^2\right] dx$$

4. Let R be the entire region enclosed by $y = x^2$ and $y = 2 - x^2$ in the upper half plane. Sketch the region. Rotate R about the x-axis and find the resulting volume. $\left[\frac{16}{3}\pi\right]$



$$V = \pi \int_{-1}^{1} [(2 - x^2)^2 - (x^2)^2] dx$$

5. S is the region enclosed by $y = -x^2 + 2x + 3$ and $y = 3x - 3$, the y=axis, and the x-axis in the first quadrant. Rotate S about the x-axis and find the volume. $\left[\frac{361\pi}{15}\right]$



$$V = \pi \int_0^1 [-x^2 + 2x + 3]^2 dx + \pi \int_1^2 [(-x^2 + 2x + 3)^2 - (3x - 3)^2] dx$$

Volume of Revolution about the x-axis-solutions

1. a) $y = x^{\frac{3}{2}}$ $\quad x=1$ & $x=3$

$$V = \pi \int_1^3 (x^{\frac{3}{2}})^2 dx$$

$$= \pi \int_1^3 x^3 \, dx$$

$$= \pi \left[\frac{x^4}{4} \right]_1^3$$

$$= \pi \left[\frac{(3)^4}{4} - \frac{(1)^4}{4} \right]$$

$$= 20\pi$$

b) $x^2 + y^2 = 16$ $\quad x=-1$ & $x=1$

$$y = \pm\sqrt{16 - x^2}$$

$$V = \pi \int_{-1}^{1} (\pm\sqrt{16-x^2})^2 dx$$

$$= \pi \int_{-1}^{1} (16 - x^2) dx$$

$$= \pi \left[16x - \frac{x^3}{3} \right]_{-1}^{1}$$

$$= \pi \left[\left(16(1) - \frac{1}{3}\right) - \left(16(-1) + \frac{1}{3}\right) \right]$$

$$= \pi \left[\frac{47}{3} + \frac{47}{3} \right]$$

$$= \frac{94\pi}{3}$$

c) $y = (2+x)^2$ $\quad x=0$ & $x=1$

$$V = \pi \int_0^1 (2+x)^4 dx$$

1 2 1
1 3 3 1
1 4 6 4 1

$$= \pi \int_0^1 \left[16 + 4(2)^3 x + 6(2)^2 x^2 + 4(2) x^3 + x^4 \right] dx$$

$$= \pi \int_0^1 (16 + 32x + 24x^2 + 8x^3 + x^4) dx$$

$$= \pi \left[16x + 16x^2 + 8x^3 + 2x^2 + \frac{x^5}{5} \right]_0^1$$

$$= \pi \left(16 + 16 + 8 + 2 + \frac{1}{5} \right)$$

$$= \frac{211\pi}{5}$$

2. Int. pts. $\sqrt{x} = 6 - x$

$$x = 36 - 12x + x^2$$

$$x^2 - 13x + 36 = 0$$

$$(x-9)(x-4) = 0$$

$\cancel{x=9}$ or $x = 4$
extraneous

$$V = \pi \int_0^4 (\sqrt{x})^2 dx + \pi \int_4^6 (6-x)^2 dx$$

$$= \pi \left[\frac{x^2}{2} \right]_0^4 + \pi \left[36x - 6x^2 + \frac{x^3}{3} \right]_4^6$$

$$= 8\pi + \pi \left[72 - \frac{208}{3} \right]$$

$$= 8\pi + \frac{8\pi}{3}$$

$$= \frac{32\pi}{3}$$

3. Int. pts. $\sqrt{x-1} = x-7$

$x-1 = x^2-14x+49$

$x^2-15x+50=0$

$(x-5)(x-10)=0$

$x=\cancel{5}$ or $x=10$

extraneous

Check for zeros: $y=x-7$ x-int: $x=7$

$y=\sqrt{x-1}$ $x=1$

$V=\pi\int_1^7 (\sqrt{x-1})^2 dx + \pi\int_7^{10} [(\sqrt{x-1})^2-(x-7)^2]dx$

$=\pi\int_1^7 (x-1)dx + \pi\int_7^{10} (-x^2+15x-50)dx$

$=\pi\left[\frac{x^2}{2}-x\right]_1^7 + \pi\left[-\frac{x^3}{3}+\frac{15}{2}x^2-50x\right]_7^{10}$

$=\pi\left[\frac{35}{2}+\frac{1}{2}\right]+\pi\left[-\frac{250}{3}+96.83\right]$

$=18\pi+13.5\pi$

$=31.5\pi$

4. Int pts. $x^2=2-x^2$

$2x^2-2=0$

$2(x^2-1)=0$

$2(x+1)(x-1)=0$

$x=\pm 1$

$V=\pi\int_{-1}^1 [(2-x^2)^2-(x^2)^2]dx$

$=\pi\int_{-1}^1 [4-4x^2+x^4-x^4]dx$

$=\pi\left[4x-\frac{4}{3}x^3\right]_{-1}^1$

$=\pi\left[\frac{8}{3}+\frac{8}{3}\right]$

$=\frac{16\pi}{3}$

5. Int. pts. $-x^2+2x+3=3x-3$

$x^2+x-6=0$

$(x+3)(x-2)=0$

$x=\cancel{-3}$ or $x=2$

not in domain

zeros: $y=3x-3$ $x=1$

$V=\pi\int_0^1 [-x^2+2x+3]^2 dx+\pi\int_1^2[[-x^2+2x+3]^2-[3x-3]^2]dx$

$=\pi\int_0^1 (x^4-2x^3-3x^2-2x^3+4x^2+6x-3x^2+6x+9)dx +$

$\pi\int_1^2 (x^4-4x^3-2x^2+12x+9-9x^2+18x-9)dx$

$=\pi\int_0^1(x^4-4x^3-2x^2+12x+9)dx+\pi\int_1^2(x^4-4x^3-11x^2+30x)dx$

$=\pi\left[\frac{x^5}{5}-x^4-\frac{2}{3}x^3+6x^2+9x\right]_0^1+\pi\left[\frac{x^5}{5}-x^4-\frac{11}{3}+15x^2\right]_1^2$

$=\frac{203\pi}{15}+\pi[10.533]$

$\doteq 24.1\pi$

Review Warm Up

1. Determine the following indefinite integrals.

   a) $\int 7x^9 dx$ b) $\int \sqrt{5x^7} dx$ c) $\int \cos(4x-1) dx$

   d) $\int \frac{x-3}{x^2-6x+5} dx$ e) $\int e^{\cos(3x)} \sin(3x)\, dx$

2. Evaluate the following definite integrals.

   a) $\int_0^2 \frac{5}{(2x+1)^2} dx$ b) $\int_0^3 x\sqrt{25-x^2} dx$

3. Find the equation of the curve $y = f(x)$ whose tangent line has a slope of $\frac{dy}{dx} = x - \sin(\pi x)$ if the curve passes through the point $\left(1, \frac{1}{2}\right)$.

4. Find the area, A, of the region bounded by the graph of $f(x) = e^{-2x}$ and the x-axis, between $x = \ln(2)$ and $x = \ln(3)$.

5. The region enclosed by the graph of the function $f(x) = \sqrt{x}\sin(x)$ between $x = 0$ and $x = \pi$ is rotated through 2π radians about the x-axis to form a solid of revolution.

   a) Show that $\frac{d}{dx}\left(\frac{1}{4}x^2 - \frac{1}{4}xsin(2x) - \frac{1}{8}\cos(2x)\right) = xsin^2(x)$

   b) Hence, show that the volume of this solid of revolution is $\frac{\pi^3}{4}$ units^3.

Review Warm Up - SOLUTIONS

1. Determine the following indefinite integrals.

a) $\int 7x^9 dx$

$= \frac{7x^{10}}{10} + C$

b) $\int \sqrt{5x^7} dx$

$= \frac{\sqrt{5}\, x^{\frac{9}{2}}}{\frac{9}{2}} + C$

$= \frac{2\sqrt{5}\, x^{\frac{9}{2}}}{9} + C$

c) $\int \cos(4x-1) dx$

$= \frac{\sin(4x-1)}{4} + C$

d) $\int \frac{x-3}{x^2-6x+5} dx$

$= \frac{\ln|x^2-6x+5|}{2} + C$

e) $\int e^{\cos(3x)} \sin(3x)\, dx$

$= \frac{-e^{\cos(3x)}}{3} + C$

2. Evaluate the following definite integrals.

a) $\int_0^2 \frac{5}{(2x+1)^2} dx$

$= \frac{-5(2x+1)^{-1}}{2} \Big]_0^2$

$= \left[\frac{-5}{2[2(2)+1]^2}\right] - \left[\frac{-5}{2[2(0)+1]^2}\right]$

$= \frac{-5}{10} + \frac{5}{2}$

$= 2$

b) $\int_0^3 x\sqrt{25-x^2} dx$

$= \frac{-2(25-x^2)^{\frac{3}{2}}}{3(2)} \Big]_0^3$

$= \frac{-(25-x^2)^{\frac{3}{2}}}{3} \Big]_0^3$

$= \frac{-[25-(3)^2]^{\frac{3}{2}}}{3} - \frac{-[25-(0)^2]^{\frac{3}{2}}}{3}$

$= \frac{-64}{3} + \frac{125}{3}$

$= \frac{61}{3}$

3. Find the equation of the curve $y = f(x)$ whose tangent line has a slope of $\frac{dy}{dx} = x - \sin(\pi x)$ if the curve passes through the point $\left(1, \frac{1}{2}\right)$.

$\because \frac{dy}{dx} = x - \sin(\pi x)$

Integrate both sides:

$y = \frac{x^2}{2} + \frac{\cos(\pi x)}{\pi} + C$

At $\left(1, \frac{1}{2}\right)$: $\frac{1}{2} = \frac{(1)^2}{2} + \frac{\cos(\pi)}{\pi} + C$

$C = \frac{1}{\pi}$

$\therefore$ The equation of the curve is $y = \frac{x^2}{2} + \frac{\cos(\pi x)}{\pi} + \frac{1}{\pi}$

4. Find the area, A, of the region bounded by the graph of $f(x) = e^{-2x}$ and the x-axis, between $x = \ln(2)$ and $x = \ln(3)$.

$\because A'(x) = e^{-2x}$

$A(x) = \frac{-e^{-2x}}{2} + C$

$A(\ln(3)) = F(\ln(3)) - F(\ln(2))$

$= \left(\frac{-e^{-2\ln(3)}}{2}\right) - \left(\frac{-e^{-2\ln(2)}}{2}\right)$

$= \left(\frac{-e^{\ln(3)^{-2}}}{2}\right) - \left(\frac{-e^{\ln(2)^{-2}}}{2}\right)$

$= -\frac{1}{2}\left(\frac{1}{9}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)$

$= \frac{5}{72}$ units^2

5. The region enclosed by the graph of the function $f(x) = \sqrt{x}\sin(x)$ between $x = 0$ and $x = \pi$ is rotated through 2π radians about the x-axis to form a solid of revolution.

a) Show that $\frac{d}{dx}\left(\frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x)\right) = x\sin^2(x)$

b) Hence, show that the volume of this solid of revolution is $\frac{\pi^3}{4}$ units^3.

a) $\frac{d}{dx}\left[\frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x)\right]$

$= \frac{x}{2} - \frac{1}{4}\left[\sin(2x) + 2x\cos(2x)\right] - \frac{1}{8}\left[-2\sin(2x)\right]$

$= \frac{x}{2} - \frac{\sin(2x)}{4} - \frac{x\cos(2x)}{2} + \frac{\sin(2x)}{4}$

$= \frac{x}{2}\left[1 - \cos(2x)\right]$

$= \frac{x}{2}\left[1 - (1 - 2\sin^2(x))\right]$

$= x\sin^2(x)$

b) $V = \pi \int_0^\pi \left[x\sin^2(x)\right] dx$

$= \pi\left[\frac{1}{4}x^2 - \frac{1}{4}x\sin(2x) - \frac{1}{8}\cos(2x)\right]_0^\pi$

$= \pi\left[\left(\frac{1}{4}(\pi)^2 - \frac{1}{4}(\pi)\sin(2\pi) - \frac{1}{8}\cos(2\pi)\right) - \left(\frac{1}{4}(0)^2 - \frac{1}{4}(0)\sin(0) - \frac{1}{8}\cos(0)\right)\right]$

$= \frac{\pi^3}{4} - 0 - \frac{1}{8} + \frac{1}{8}$

$= \frac{\pi^3}{4}$ units^3

MCV4UE Relax…..Review

Note: #2, 6, 12, 13d, 17 and 20 are not part of the course.

Integral the following

1) $\int \frac{x^3}{\sqrt{1+x^4}}dx$

2) $\int \cot^3(x)\csc^2(x)dx$

3) $\int \cos(x)(\sin(x)+7)^2 dx$

4) $\int \frac{x+1}{(x^2+2x-3)^2}dx$

5) $\int \tan(x)dx$

6) $\int \frac{\cos(x)}{1+\sin^2(x)}dx$

7) $\int 3xe^{x^2}dx$

8) $\int \left(\frac{x^2-6x+3}{\sqrt{x}}\right)dx$

9) $\int \frac{\ln(2x)}{x}dx$

10) $\int \frac{\sec(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}dx$

11) $\int \frac{e^{2x}}{1-e^{2x}}dx$

12) Solve the initial value problem. $\frac{dy}{dx} = 6xe^{x^2} + y^2 6xe^{x^2}$, $y(0)=0$

13) Evaluate the following definite integrals ***without*** the use of your calculator.

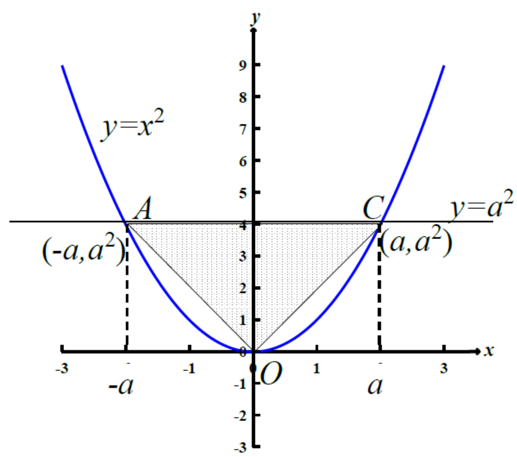
a) $\int_{-1}^{4} e^{3x}dx$

b) $\int_0^{\pi}\left(\cos 2x + 6x^2 - \frac{5}{x+1}\right)dx$

c) $\int_1^{e}\left(\frac{x^2-1}{x}\right)dx$

d) $\int_{-5}^{5}|10-x|\,dx$

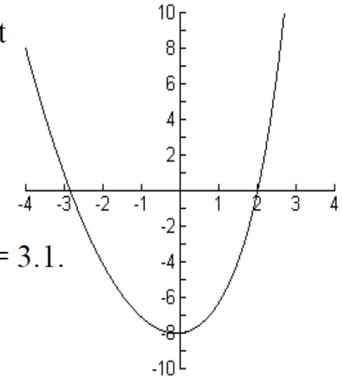
14) Figure below shows triangle *AOC* inscribed in the region cut from the parabola $y=x^2$ by the line $y=a^2$. Find the ratio of the area of the triangle to the area of parabolic region.



15) The function *f*, defined by $f(t) = 6+\cos\left(\frac{t}{10}\right)+3\sin\left(\frac{7t}{40}\right)$, is used to model the acceleration of a plane, in miles per square minute, for $0 \le t \le 40$. According to this model, what is the velocity of the plane at $t = 23$, if $v(0) = 7\,mile/\min$?

16) Find the area of the region bounded by the curves $y = \cos(2x)$, $y = \sin(x)$, $x = -\frac{\pi}{2}$ and $x = \frac{5\pi}{6}$.

17) (Calculator) Let $g(x) = 2xe^{(x-2)} + x^2 - 8$ as shown at right. Let f represent the position of a particle on a number line. Define f as follows: $s(t) = 5 + \int_3^t g(x)dx$ for time $0 \le t \le 8$.



(a) When is the particle moving to the left?
(b) Write the equation for the line tangent to the graph of s at time $t = 3$.
(c) Use your tangent line to approximate the position of the particle at time $t = 3.1$.
(d) Where is the particle at time $t = 7$?
(e) How far has the particle travelled in the first 7 seconds?

18) Let f be a function with the following properties:

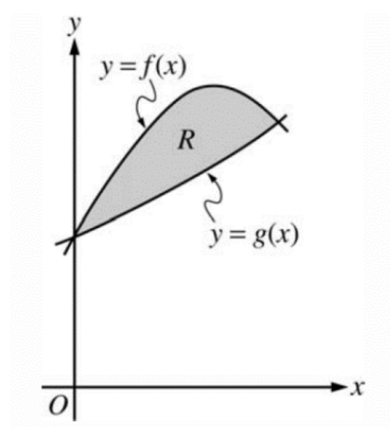
i) $f'(x) = ax^2 + bx$ ii) $f'(1) = 6$ iii) $f''(1) = 18$ iv) $\int_1^2 f(x)dx = 18$

Find an algebraic expression for $f(x)$.

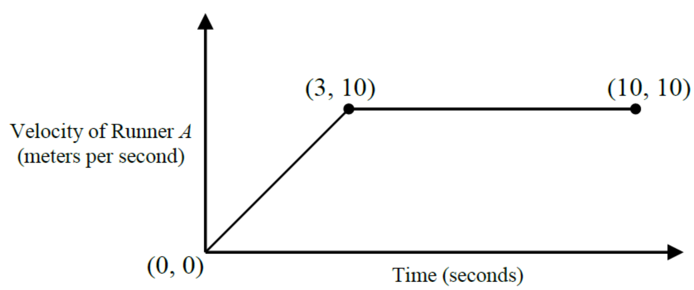
19) The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find its volume.

20) Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure below.

(a) Find the area of R.
(b) Find the volume of the solid generated when R is revolved about the x-axis.
(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of this solid.



21) Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph below, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \frac{24t}{2t+3}$.



(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.
(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

MCV4UE Relax…..Review **SOLUTION**

Integral the following

1) $\int \frac{x^3}{\sqrt{1+x^4}}dx$

$1+x^4 = u$

$4x^3 dx = du$

$x^3 dx = \frac{du}{4}$

$\int \frac{x^3}{\sqrt{1+x^4}}dx = \frac{1}{4}\int u^{-\frac{1}{2}} du$

$= \frac{1}{2}\left(1+x^4\right)^{\frac{1}{2}} + c$

Don't do ~~2)~~ $\int \cot^3(x)\csc^2(x)dx$

$I = \int\left[\csc^2(x)-1\right]\csc(x)\left[\cot(x)\csc(x)\right]dx$

$\csc(x) = u$

$-\cot(x)\csc(x)dx = du$

$I = \int\left(u^2-1\right)u\,du$

$= \frac{1}{4}u^4 - \frac{1}{2}u^2 + c$

$= \frac{1}{4}\csc^4(x) - \frac{1}{2}\csc^2(x) + c$

3) $\int \cos(x)(\sin(x)+7)^2 dx$

$\sin(x)+7 = u$

$\cos(x)dx = du$

$I = \int u^2 du$

$= \frac{1}{3}u^3 + c$

$= \frac{1}{3}\left(\sin(x)+7\right)^3 + c$

4) $\int \frac{x+1}{(x^2+2x-3)^2}dx$

$x^2+2x-3 = u$

$(2x+2)dx = du$

$I = \frac{1}{2}\int \frac{du}{u^2}$

$= -\frac{1}{2u} + c$

$= -\frac{1}{2\left(x^2+2x-3\right)} + c$

5) $\int \tan(x)dx$

$\cos(x) = u$

$-\sin(x)dx = du$

$I = -\int \frac{du}{u}$

$= -\ln|\cos(x)| + c$

Don't do ~~6)~~ $\int \frac{\cos(x)}{1+\sin^2(x)}dx$

$\sin(x) = u$

$\cos(x)dx = du$

$I = \int \frac{du}{1+u^2}$

$= \arctan(u) + c$

$= \arctan(\sin(x)) + c$

7) $\int 3xe^{x^2}dx$

$$=\frac{3}{2}e^{x^2}+c$$

8) $\int\left(\frac{x^2-6x+3}{\sqrt{x}}\right)dx$

$$x=u^2$$
$$dx=2udu$$
$$I=2\int\left(\frac{u^4-6u^2+3}{\cancel{u}}\right)\cancel{u}du$$
$$=\frac{2}{5}u^5-4u^3+6u+c$$
$$=\frac{2}{5}x^{\frac{5}{2}}-4x^{\frac{3}{2}}+6x^{\frac{1}{2}}+c$$

9) $\int\frac{\ln(2x)}{x}dx$

$$\ln(2x)=u$$
$$\frac{dx}{x}=du$$
$$I=\int udu$$
$$=\frac{1}{2}u^2+c$$
$$=\frac{1}{2}\ln^2(2x)+c$$

10) $\int\frac{\sec(\sqrt{x})\tan(\sqrt{x})}{\sqrt{x}}dx$

$$\sec(\sqrt{x})=u$$
$$\frac{\sec(\sqrt{x})\tan(\sqrt{x})}{2\sqrt{x}}dx=du$$
$$I=2\int du$$
$$=2u+c$$
$$=2\sec(\sqrt{x})+c$$

11) $\int\frac{e^{2x}}{1-e^{2x}}dx$

$$1-e^{2x}=u$$
$$-2e^{2x}dx=du$$
$$I=\frac{-1}{2}\int\frac{du}{u}$$
$$=\frac{-1}{2}\ln\left|1-e^{2x}\right|+c$$

Don't do

~~12)~~ Solve the initial value problem. $\frac{dy}{dx} = 6xe^{x^2} + y^2 6xe^{x^2}$, $y(0) = 0$

$$\frac{dy}{dx} = 6xe^{x^2}\left(1+y^2\right)$$

$$\frac{dy}{\left(1+y^2\right)} = 6xe^{x^2}dx$$

$$\int \frac{dy}{\left(1+y^2\right)} = \int 6xe^{x^2}dx$$

$$\arctan(y) = 3e^{x^2} + c \leftarrow \text{sub. in } (0,0)$$

$$0 = 3 + c$$

$$c = -3$$

$$\arctan(y) = 3e^{x^2} - 3$$

$$y = \tan\left(3e^{x^2} - 3\right)$$

13) Evaluate the following definite integrals ***without*** the use of your calculator.

a) $\int_{-1}^{4} e^{3x}dx$

$$= \frac{1}{3}e^{3x}\Big]_{-1}^{4}$$

$$= \frac{1}{3}\left(e^{12} - e^{-3}\right)$$

b) $\int_0^{\pi}\left(\cos 2x + 6x^2 - \frac{5}{x+1}\right)dx$

$$= \frac{1}{2}\sin(2x) + 3x^2 - 5\ln|x+1|\Big]_0^{\pi}$$

$$= 2\pi^3 - 5\ln(\pi+1)$$

c) $\int_1^e\left(\frac{x^2-1}{x}\right)dx$

$$= \int_1^e x\,dx - \int_1^e \frac{dx}{x}$$

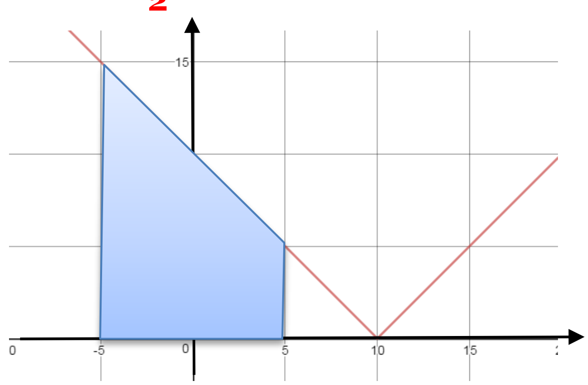
$$= \frac{1}{2}x^2\Big]_1^e - \ln(x)]_1^e$$

$$= \frac{1}{2}e^2 - \frac{3}{2}$$

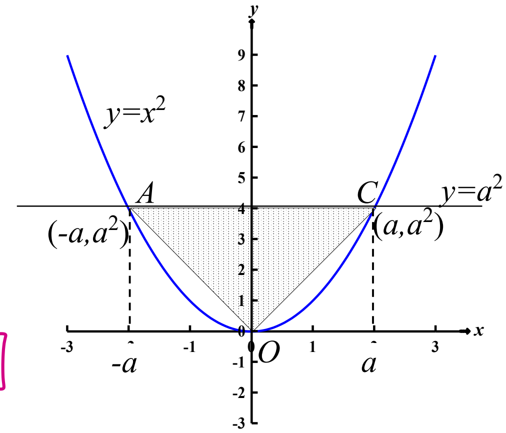
Don't do

~~d)~~ $\int_{-5}^{5}|10-x|\,dx = \underbrace{\frac{1}{2}(10)(15+5)}_{\text{Area of trapezoid}} = 100$

Area of trapezoid



14) Figure below shows triangle AOC inscribed in the region cut from the parabola $y=x^2$ by the line $y=a^2$. Find the ratio of the area of the triangle to the area of parabolic region.

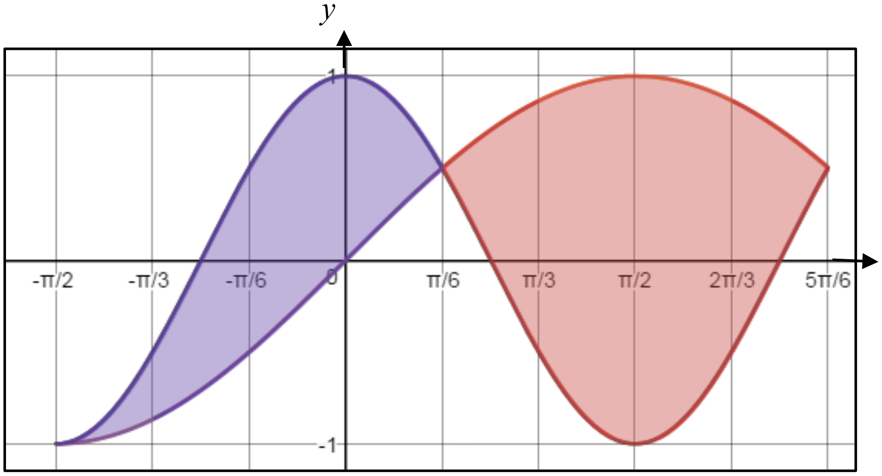


$$A_\Delta = \frac{1}{2}(2a)(a^2)$$
$$= a^3$$
$$A = 2\int_0^a (a^2 - x^2)\,dx \quad \text{or} \quad A = \int_{-a}^{a} (a^2 - x^2)\,dx$$
$$= 2\left[a^2x - \frac{x^3}{3}\right]_0^a \qquad = \left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$
$$= 2\left(a^3 - \frac{a^3}{3}\right) \qquad = \left[a^2(a) - \frac{(a)^3}{3}\right] - \left[a^2(-a) - \frac{(-a)^3}{3}\right]$$
$$= a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}$$
$$= \frac{4}{3}a^3 \qquad = \frac{4}{3}a^3$$
$$\frac{A_\Delta}{A} = \frac{a^3}{\frac{4}{3}a^3}$$
$$\frac{A_\Delta}{A} = \frac{3}{4}$$

15) The function f, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the acceleration of a plane, in miles per square minute, for $0 \le t \le 40$. According to this model, what is the velocity of the plane at $t = 23$, if $v(0) = 7\,mile/\min$?

$$v(t) = \int 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right) dt$$
$$v(t) = 6t + 10\sin\left(\frac{t}{10}\right) - \frac{120}{7}\cos\left(\frac{7t}{40}\right) + c$$
$$v(0) = 7 : c = \frac{169}{7}$$
$$v(t) = 6t + 10\sin\left(\frac{t}{10}\right) - \frac{120}{7}\cos\left(\frac{7t}{40}\right) + \frac{169}{7}$$
$$v(23) \doteq 180.477 \text{ mile/min}$$
$$\doteq 180 \text{ mile/min}$$

16) Find the area of the region bounded by the curves $y=\cos(2x)$, $y=\sin(x)$, $x=-\frac{\pi}{2}$ and $x=\frac{5\pi}{6}$.



$$A=\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}}[\cos(2x)-\sin(x)]dx+\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\sin(x)-\cos(2x)]dx$$

$$=\left[\frac{1}{2}\sin(2x)+\cos(x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}+\left[-\cos(x)-\frac{1}{2}\sin(2x)+\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

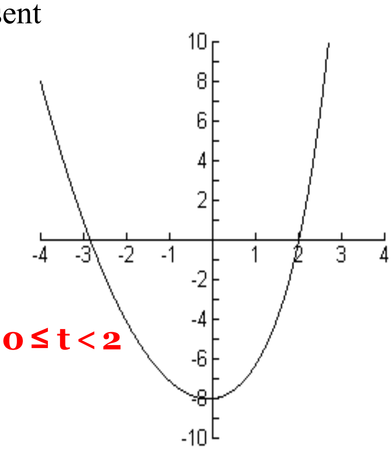
$$=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}$$

$$=\frac{9\sqrt{3}}{4}$$

Don't do

17) (Calculator) Let $g(x)=2xe^{(x-2)}+x^2-8$ as shown at right. Let s represent the position of a particle on a number line. Define s as follows:

$s(t)=5+\int_3^t g(x)dx$ for time $0\le t\le 8$.



(a) When is the particle moving to the left?

$\mathbf{v(t)=s'(t)=g(t)}$

particle is moving to the left when $v(t)<0$, i.e $g(t)<0$

the graph of g(x) lies below the x-axis for $t\ge 0$ when $0\le t<2$

(b) Write the equation for the line tangent to the graph of s at time $t=3$.

$$m_t=s'(3)$$
$$=g(3)$$
$$=2(3)e^{(3-2)}+3^2-8$$
$$=6e+1$$

$$s(3)=5+\int_3^3 g(x)dx$$
$$=5$$

Equation of the tangent line at point (3,5) is $y-5=(6e+1)(x-1)$

(c) Use your tangent line to approximate the position of the particle at time $t = 3.1$.

$$y - 5 = (6e + 1)(3.1 - 1)$$
$$y = 41.35 \text{ m to the right of origin}$$

(d) Where is the particle at time $t = 7$?

$$s(7) = 5 + \int_3^7 \left(2xe^{(x-2)} + x^2 - 8\right)dx$$
$$= 1560.418$$

(e) How far has the particle travelled in the first 7 seconds?

$$\text{distance} = \int_0^7 \left|2xe^{(x-2)} + x^2 - 8\right|dx$$
$$= 1843.971 \text{ m}$$

18) Let f be a function with the following properties:

i) $f'(x) = ax^2 + bx$ ii) $f'(1) = 6$ iii) $f''(1) = 18$ iv) $\int_1^2 f(x)dx = 18$

Find an algebraic expression for $f(x)$.

$$f'(1) = 6 : a + b = 6 \quad (1)$$
$$f(x) = \int \left(ax^2 + bx\right)dx$$
$$f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + c$$
$$f'(x) = ax^2 + bx \rightarrow f''(x) = 2ax + b$$
$$f''(1) = 18 : 2a + b = 18 \quad (2)$$
$$\begin{cases} a + b = 6 \\ 2a + b = 18 \end{cases} \Rightarrow a = 12 \ \& \ b = -6$$
$$\therefore f(x) = 4x^3 - 3x^2 + c$$
$$\because \int_1^2 \left(4x^3 - 3x^2 + c\right)dx = 18$$
$$\therefore x^4 - x^3 + cx \Big]_1^2 = 18$$
$$16 - 8 + 2c - 1 + 1 - c = 18$$
$$c = 10$$
$$\therefore f(x) = 4x^3 - 3x^2 + 10$$

19) The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find its volume.

$y = x^2 + 1$

$y = -x + 3$

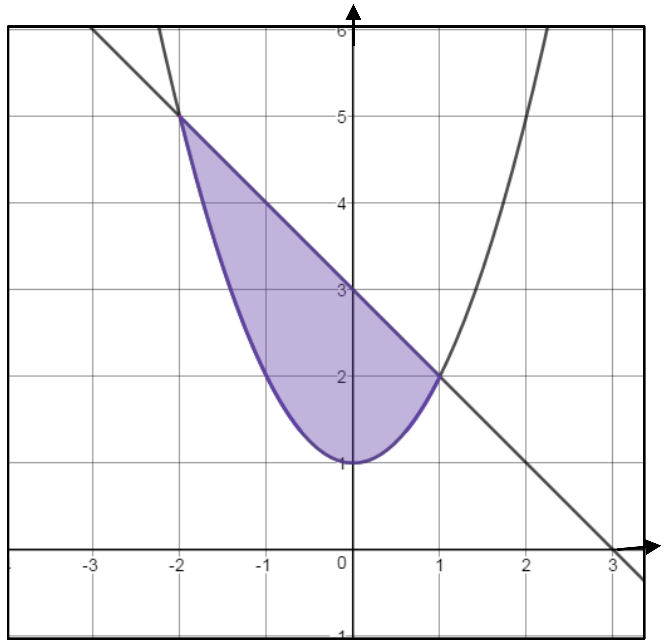
Finding point of intersection

$x^2 + 1 = -x + 3$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2\ ,\ x = 1$



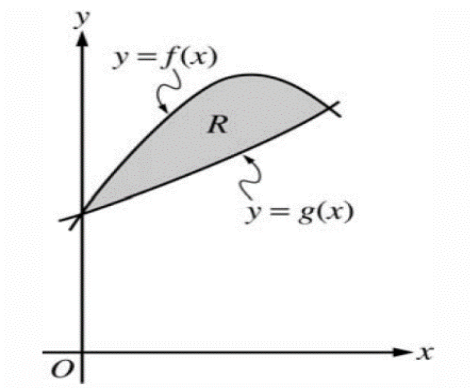
$$v = \pi\int_{-2}^{1}\left[(-x+3)^2 - (x^2+1)^2\right]dx$$

$$v = \pi\int_{-2}^{1}\left[-x^4 - x^2 - 6x + 8\right]dx$$

$$= \pi\left[\frac{-1}{5}x^5 - \frac{1}{3}x^3 - 3x^2 + 8x\right]_{-2}^{1}$$

$$= \frac{117\pi}{5}$$

Don't do 20) Let f and g be the functions given by $f(x) = 1 + \sin(2x)$ and $g(x) = e^{x/2}$. Let R be the shaded region in the first quadrant enclosed by the graphs of f and g as shown in the figure below.



(a) Find the area of R.

Point of intersection

$f(x) = 1 + \sin(2x)$

$g(x) = e^{x/2}$

$1 + \sin(2x) = e^{x/2}$ ↓using GDC

$x = 0\ ,\ x = 1.135$

Let $B = 1.135$

$$A = \int_0^B \left[1 + \sin(2x) - e^{x/2}\right]dx$$

$$= 0.4291\ u^2$$

(b) Find the volume of the solid generated when R is revolved about the x-axis.

$$V = \pi \int_0^B \left[\left(1+\sin(2x)\right)^2 - \left(e^{x/2}\right)^2 \right] dx$$

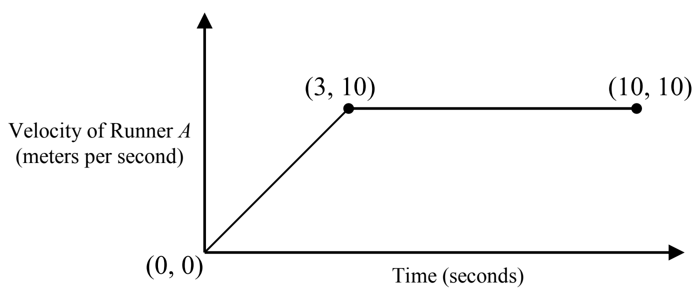
$$= 4.2665 \text{ u}^3$$

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are semicircles with diameters extending from $y = f(x)$ to $y = g(x)$. Find the volume of this solid.

$$V = \frac{\pi}{2} \int_0^B \left[\left(\frac{1+\sin(2x) - e^{x/2}}{2} \right)^2 \right] dx$$

$$= 0.077 \text{ u}^3$$

21) Two runners, A and B, run on a straight racetrack for $0 \le t \le 10$ seconds. The graph below, which consists of two line segments, shows the velocity, in meters per second, of Runner A. The velocity, in meters per second, of Runner B is given by the function v defined by $v(t) = \dfrac{24t}{2t+3}$.



(a) Find the velocity of Runner A and the velocity of Runner B at time $t = 2$ seconds. Indicate units of measure.

$$\text{Runner A : velocity} = \left(\frac{10}{3}\right) 2$$

$$= \frac{20}{3} \doteq 6.667 \text{ m/s}$$

$$\text{Runner B : } v(2) = \frac{48}{7} \doteq 6.857 \text{ m/s}$$

(b) Find the acceleration of Runner A and the acceleration of Runner B at time $t = 2$ seconds. Indicate units of measure.

$$\text{Runner A : acceleration} = \frac{10}{3} \text{ m/s}^2$$

$$\text{Runner B : } a(2) = v'(2) = \left.\frac{72}{(2t+3)^2}\right|_{t=2}$$

$$\doteq 1.469 \text{ m/s}^2$$