# Unit #3
# Intro to Vectors
# Coursepack

# Introduction to Vectors

**Scalar and Vector Quantities:**

*Scalar*

- a quantity with magnitude only.

Examples:

- ____________________ • ____________________
- ____________________ • ____________________
- ____________________ • ____________________

*Vector*

- a mathematical quantity that is expressed by a magnitude AND direction

Examples:

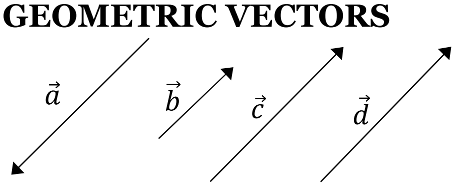
- ____________________ • ____________________
- ____________________ • ____________________
- ____________________ • ____________________

Example 1: the following situations need to be described using an appropriate measure. Classify the measure as a scalar or a vector.

(a) the cost of a dance ticket _____________
(b) the path from your desk to the classroom door
(c) the air speed of a jet as it heads north _____________________

**REPRESENTATION OF VECTORS:**

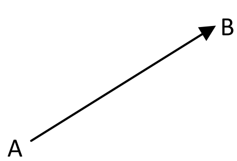
**1. GEOMETRIC VECTORS** A vector represented by a directed line segment drawn so that its length represents its magnitude.




**2. ALGEBRAIC VECTORS** A vector that is written in rectangular form.

- **RECTANGULAR FORM:** is of the form (a, b) or $\begin{pmatrix} a \\ b \end{pmatrix}$ where a represents the x-value and b represents the y-value of the terminal point of the vector. Its initial point is the origin (0, 0). They are also called **algebraic vectors** and IB uses them in the column vector form.ie. (3, 4) becomes $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

**Notation:**




$\overrightarrow{AB}$, $\vec{v}$ , **v (boldface)**

Vector $\overrightarrow{AB}$ has an initial point (tip) at A and a terminal point (tail) at B.

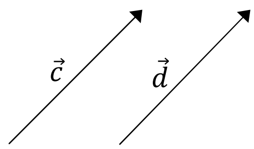
**Magnitude of a Vector**

$|\overrightarrow{\boldsymbol{AB}}|$ represents the MAGNITUDE of $\overrightarrow{\boldsymbol{AB}}$**.**

**| v |** represents the MAGNITUDE of **v.**

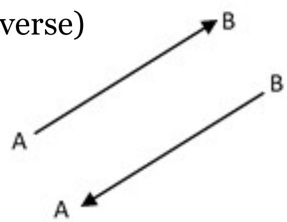
- If the magnitude of a vector is zero, we call it the zero vector and denote it $\vec{0}$ . This is a useful vector despite that its direction is undefined.

**Equality of Vectors**




Two vectors are equal if they have the same magnitude and the same direction. Ex. $\vec{c} = \vec{d}$, $|\vec{c}| = |\vec{d}|$

**Opposite Vectors**
(additive inverse)

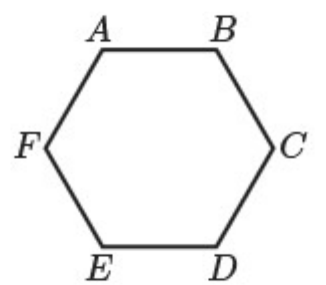



The opposite of a vector would have the same magnitude but opposite direction. Ex. $\vec{a} = -\vec{d}$, $|\vec{a}| = |\vec{d}|$

$\overrightarrow{\boldsymbol{AB}} = -\overrightarrow{\boldsymbol{BA}}$ $\quad$ ∴ **directions are opposite**

$|\overrightarrow{\boldsymbol{AB}}| = |\overrightarrow{\boldsymbol{BA}}|$ $\quad$ ∴ **magnitudes are still equal**

Example 2: *ABCDEF* is a regular hexagon. Give examples of vectors formed between pairs of vertices of hexagon *ABCDEF*:




a. equal

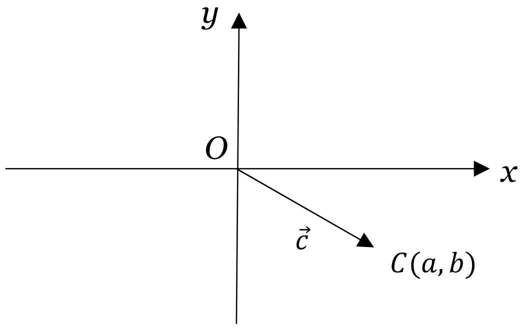
b. parallel but with different magnitudes

c. equal in magnitude but opposite in direction

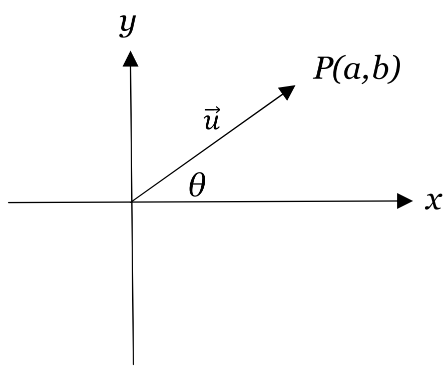
d. equal in magnitude but not parallel

e. different in both magnitude and direction

Note: vector starting from the origin O to another point C is called position vector of C (i.e., fixed with respect to the origin). $\overrightarrow{OC} = \vec{c}$ or $\overrightarrow{OC} = \binom{a}{b}$ 'column vector'

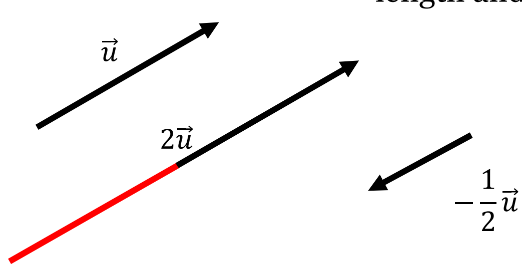



**Cartesian Co-ordinate System**: $\vec{u}$ can be represented as an ordered pair (a, b) where its magnitude (modulus) is $|\vec{u}| = \sqrt{a^2 + b^2}$ and direction $\theta = \tan^{-1}\binom{a}{b}$ with $\theta$ measured counter-clockwise from the positive $x$-axis to the line of the vector. The ordered pair (a, b) is referred to as an **algebraic vector.** The values of $a$ and $b$ are the $x$- and $y$-components of the vector.




**Scalar Multiplication:**

Multiplying a vector by a scalar will change its length and/or its direction




**Collinear Vectors**

- vectors that lie on the same line when they are in standard position.
- they will be parallel to each other.
- they either have the same direction or opposite direction.
- one will always be a scalar multiple of the other. i.e. $\boldsymbol{\vec{u} = k\vec{v}, k \in R, k \neq 0}$

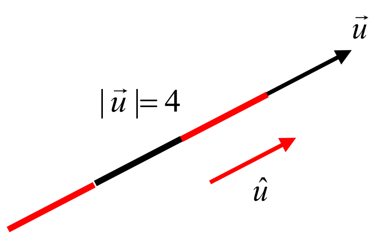
Example 3. Given that $|\vec{u}| = 5$ find the magnitude of each of the following vectors:

a. $2\vec{u}$ b. $-4\vec{u}$ c. $\frac{1}{5}\vec{u}$

Example 4. Determine the value of k so that the following pairs of vectors are collinear.

a. $\begin{pmatrix} 5 \\ -7 \end{pmatrix} = k\begin{pmatrix} -10 \\ 14 \end{pmatrix}$ b. $\vec{v} = \left(1, \frac{k}{2} - 6\right)$ and $\vec{u} = \left(-4, 1 + \frac{k}{6}\right)$

**The Unit Vector:**




A unit vector is a vector 1 unit long.

$\hat{u}$ is a unit vector parallel to $\vec{u}$

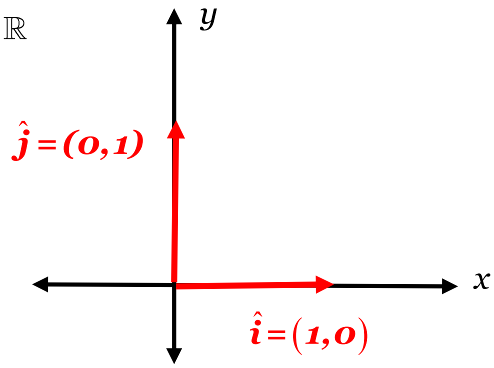
$$\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$$

To create a unit vector in the direction of a non-zero $\vec{u}$, multiply $\vec{u}$ by the scalar equal to the reciprocal of the magnitude of $\vec{u}$.

Example 5: If $|\vec{a}| = 12$, state a unit vector in the opposite direction of $\vec{a}$.

**Standard Unit Vectors:** The special unit vectors which point in the direction of the positive x-axis and positive y-axis are given the names $\hat{i}$ and $\hat{j}$ respectively, where $\hat{i}=(1,0)$ and $\hat{j}=(0,1)$.

$\hat{i}$ and $\hat{j}$ are the ***standard basis vectors*** for $\mathbb{R}^2=\mathbb{R}\times\mathbb{R}$

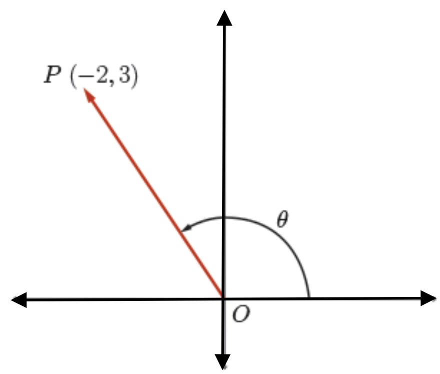



We may express any vector in the *xy*-plane as a sum of scalar multiples of the vectors and.

$\vec{u}=\overrightarrow{OP}=(a,b)$ or $\vec{u}=a\hat{i}+b\hat{j}$ or $\vec{u}=\begin{pmatrix}a\\b\end{pmatrix}$ , $|\vec{u}|=\sqrt{a^2+b^2}$

Example 6 : $\overrightarrow{OP}=(-2,3)=-2\hat{i}+3\hat{j}$

$$|\overrightarrow{OP}|=\sqrt{(-2)^2+(3)^2}$$
$$=\sqrt{13}$$
$$\tan(\theta)=\frac{3}{-2}\Rightarrow \alpha=\tan^{-1}\left(\frac{3}{2}\right)\approx 56.3^0$$
$$\theta\approx 180^0-56.3^0\approx 123.7^0$$




Example 7: Find a vector of magnitude $\sqrt{6}$ in the direction of $\vec{v}=7\hat{i}+5\hat{j}$ .

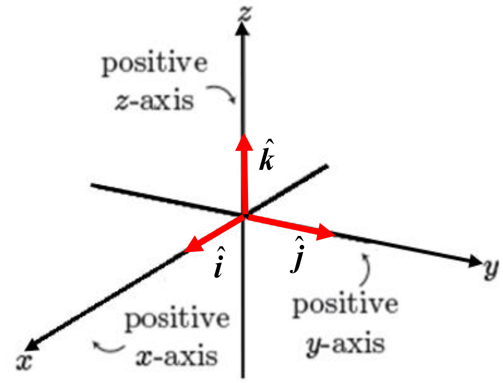
**Vectors in 3 Dimensions**

Previously, we had considered geometric and algebraic vectors in the 2-dimensional (Cartesian) plane. This model extends naturally to 3 dimensions.

Consider the 2-D (*xy*) Cartesian plane comprised of the *x* and *y*-axes.

If we add a third axis (*z*-axis) to our existing *xy*-plane such that all 3 axes are mutually perpendicular to one another, we create a coordinate system which models 3-dimensional space.

To plot the 3-dimensional point with coordinates (*a*,*b*,*c*), move *a* units from the origin in the *x*-direction, *b* units in the *y*-direction, and *c* units in the *z*-direction.

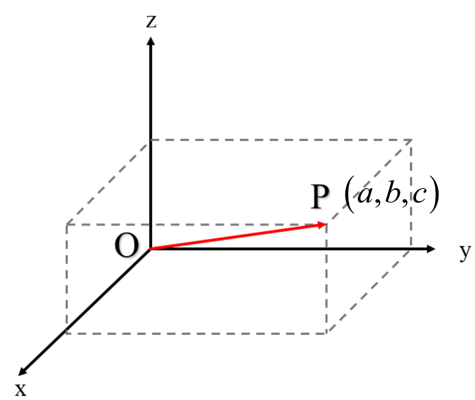



**Standard Unit Vectors in R3:** $\hat{i}=(1,0,0), \hat{j}=(0,1,0)$, and $\hat{k}=(0,0,1)$ are the special unit vectors pointing in the direction of the positive x-, y-, and z-axes, respectively.

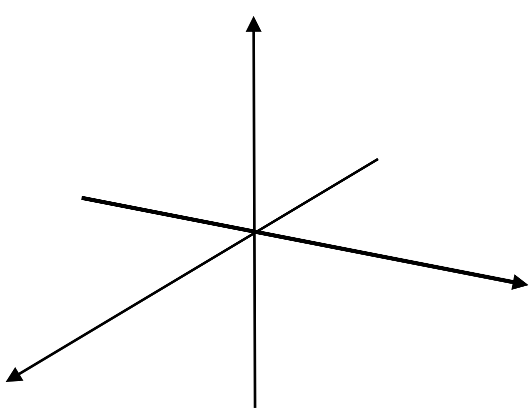
$$\overrightarrow{OP}=a\hat{i}+b\hat{j}+c\hat{k}$$

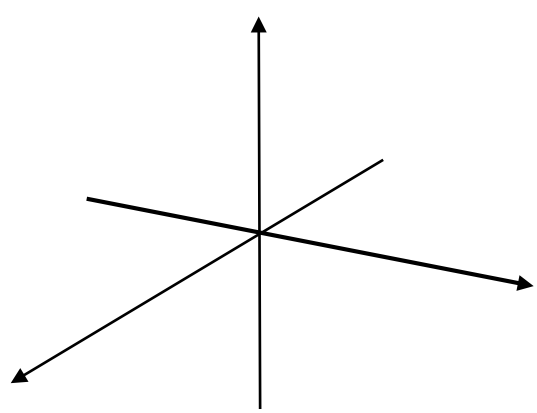
$$\overrightarrow{OP}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$|\overrightarrow{OP}|=\sqrt{a^2+b^2+c^2}$$

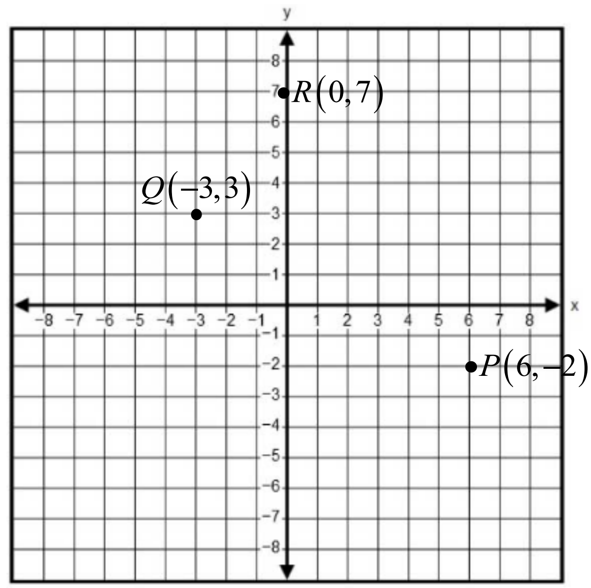



Example 8: Given the coordinates of points P(2,4,5) and Q(-4,3,-2), draw the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ using a rectangular prism.



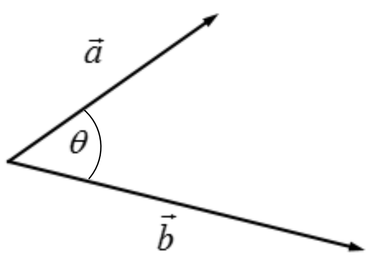


Example 9: Express the position vector of each of the points shown in the diagram as an ordered pair, column vector, and in basis vector notation.




**Angle between 2 vectors:**

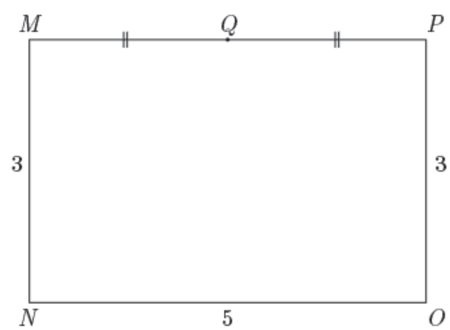
The angle between two vectors is the angle $\leq 180°$ formed when the vectors are placed **tail to tail**, that is, starting at the same point.




Example 10: *MNOP* is a rectangle with side lengths 3 and 5.

*Q* is the midpoint of *MP*.

Find the angle between the following vectors:



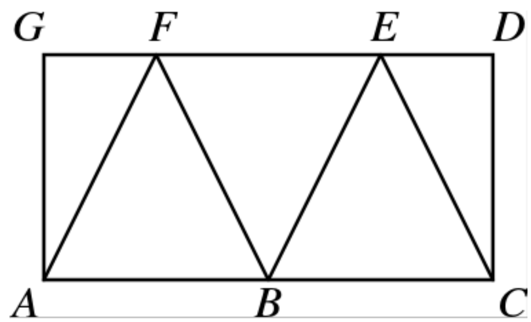

a) $\overrightarrow{NP}$ and $\overrightarrow{NO}$

b) $\overrightarrow{NM}$ and $\overrightarrow{NQ}$

c) $\overrightarrow{NQ}$ and $\overrightarrow{QP}$

**Practice**

1. In the diagram at the right, △ AF B and △ BEC are equilateral, and ACDG is a rectangle.




(a) Write down two other vectors **equal** to $\overrightarrow{AB}$.

(b) Write down three vectors which are **opposite** to $\overrightarrow{FE}$

(c) What vector is the **opposite** of $\overrightarrow{DC}$ ?

(d) Write down 3 vectors which have the same magnitude as $\overrightarrow{BC}$, but different direction.

(e) What vector is equal to 2 $\overrightarrow{FE}$ ?

(f) What vector is equal to $\frac{1}{2}\overrightarrow{FE}$ ?

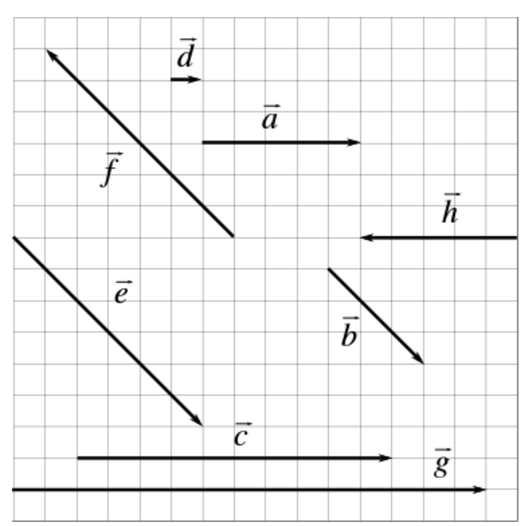
2. Using the diagram from #1, find the angles between the following vectors:

(a) $\overrightarrow{AB}$ and $\overrightarrow{AF}$
(b) $\overrightarrow{AF}$ and $\overrightarrow{AG}$
(c) $\overrightarrow{DC}$ and $\overrightarrow{AB}$
(d) $\overrightarrow{BC}$ and $\overrightarrow{CE}$
(e) $\overrightarrow{EC}$ and $\overrightarrow{AG}$
(f) $\overrightarrow{FD}$ and $\overrightarrow{BA}$

3. Sketch a vector to represent each of the following quantities, using the specified scale:

(a) a velocity of 30 m/s [south], where 1 cm = 10 m/s.
(b) a force of 20 Newtons, straight down, where 1 cm = 10 N .
(c) a displacement of 25 metres to the right, where 1 cm = 10 m.
(d) an airplane taking off a runway at an angle of 30° at a speed of 40 km/h, where 1 cm =10 km/h.

4. Using the grid at the right, choose a vector which equals:




(a) $-\vec{a}$
(b) $3\vec{a}$
(c) $-2\vec{b}$
(d) a unit vector parallel to $\vec{a}$

5. Given the vector $\vec{u}$ such that $|\vec{u}| = 8$ units, find the following:

(a) $|3\vec{u}|$
(b) $\left|-\frac{3}{4}\vec{u}\right|$
(c) $|-7\vec{u}|$

6. Determine a unit vector parallel to each of the following vectors:
   (a) $\vec{a}$, given that $|\vec{a}| = 12$ units
   (b) $\vec{w}$, given that $|\vec{w}| = 10$ units
   (c) $\vec{u}$ (non-zero)

7. A boat leaves harbour at 2:00 and travels due south at 50 km/h until 3: 30, when it turns east and travels at the same speed for another hour.

   (a) Write down the displacement vectors for each part of the journey.
   (b) What is the total distance covered?
   (c) What is the displacement vector between the starting point and ending point?

8. Two planes leave an airport at the same time. Plane A travels northwest at 120 km/h, while plane B travels due east at 150 km/h. After one hour, they both land. If plane A must then travel to plane B's landing point, in what direction should it travel, and how long will it take if it travels at 120 km/h?

9. For each point Q given, write the position vector $\overrightarrow{OQ}$ in terms of $\hat{i}$ and $\hat{j}$ .

a. Q (3, −4)

b. Q (−5, −1)

10. For each point Q in question 1, find the magnitude of the position vector $\overrightarrow{OQ}$ and its direction relative to the positive x-axis.

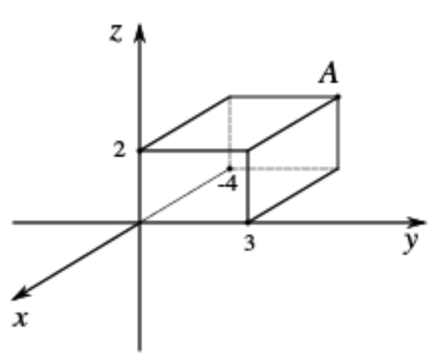
11. For each point R given, find the magnitude of the position vector $\overrightarrow{OR}$ .

a. R (4, −3, 12)

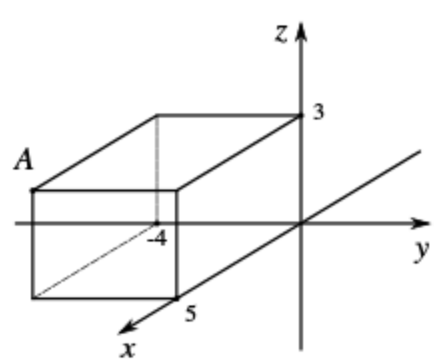
b. R (2, −1, 3)

12. Write the position vectors of the point A shown, in the form $a\hat{i} + b\hat{j} + c\hat{k}$ .

a.




b.



13. Draw a sketch to show the point D (4, 2, −3) and draw the position vector $\overrightarrow{OD}$.

14. Determine the direction angles for each of the following vectors.

a. $\vec{v} = 2\hat{i} - \hat{j} + 3\hat{k}$

b. $\overrightarrow{OA} = (-1, 4, -5)$

c. $\vec{u} = 5\hat{i} - 12\hat{k}$

d. $\overrightarrow{OB} = (0, 3, -4)$

15. Find a unit vector parallel to each of the given vectors.

a. $\vec{v} = (2, -5)$

b. $\overrightarrow{OZ} = \hat{i} - 2\hat{j} + 4\hat{k}$

c. $\vec{w} = (-5, 12)$

d. $\overrightarrow{OP} = 3\hat{i} + 3\hat{j} - \hat{k}$

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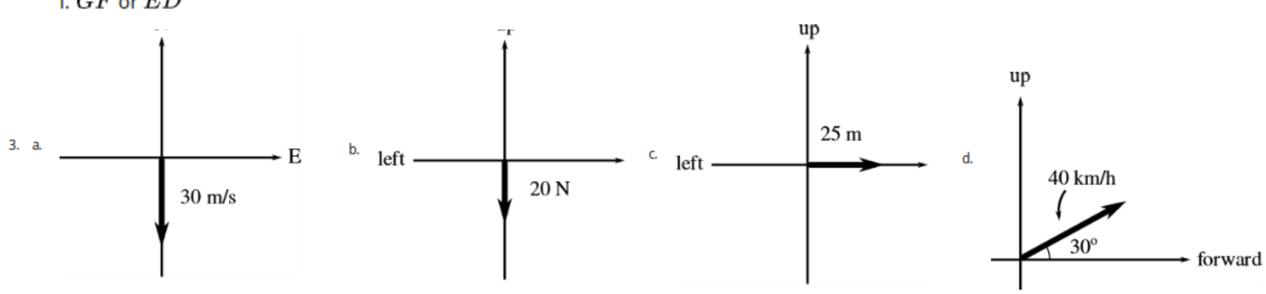
Answers

1. a. $\overrightarrow{BC}$ and $\overrightarrow{FE}$
   b. $\overrightarrow{BA}, \overrightarrow{CB}$ and $\overrightarrow{EF}$
   c. $\overrightarrow{CD}$ or $\overrightarrow{AG}$
   d. Any of $\overrightarrow{CB}, \overrightarrow{BA}, \overrightarrow{EF}, \pm\overrightarrow{AF}, \pm\overrightarrow{FB}, \pm\overrightarrow{BE}, \pm\overrightarrow{CE}$
   e. $\overrightarrow{AC}$
   f. $\overrightarrow{GF}$ or $\overrightarrow{ED}$

2. a. $60^\circ$
   b. $30^\circ$
   c. $90^\circ$
   d. $120^\circ$
   e. $150^\circ$
   f. $180^\circ$




4. a. $\vec{h}$
   b. $\vec{g}$
   c. $\vec{f}$
   d. $\vec{d}$

5. a. 24
   b. 6
   c. 56

6. a. $a = \dfrac{\vec{a}}{12}$
   b. $w = \dfrac{\vec{w}}{10}$
   c. $u = \dfrac{\vec{u}}{|\vec{u}|}$

7. a. $75\ \text{km}[S]$, $50\ \text{km}[E]$
   b. $125\ \text{km}$
   c. $90.1\ \text{km}[S33.7^\circ E]$

8. $[E\ 20^\circ\ S]$ for 2 hours and 5 minutes

9. a. $3\hat{i} - 4\hat{j}$

   b. $-5\hat{i} - \hat{j}$

10. a. $\left|\overrightarrow{OQ}\right| = 5, 307°$ or $-53°$ from the positive $x$ axis.

    b. $\left|\overrightarrow{OQ}\right| = \sqrt{26}, 191°$ or $-169°$ from the positive $x$-axis.

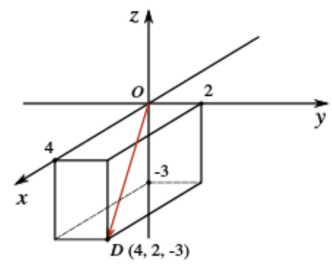
11. a. $\left|\overrightarrow{OR}\right| = 13$

    b. $\left|\overrightarrow{OR}\right| = \sqrt{14}$

12. a. $\overrightarrow{OA} = -4\hat{i} + 3\hat{j} + 2\hat{k}$

    b. $\overrightarrow{OA} = 5\hat{i} - 4\hat{j} + 3\hat{k}$

13. Answers may vary.




14. $\vec{v} \approx 5\hat{i} - 2\hat{j} - 4\hat{k}$

15. a. $\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right)$

    b. $\frac{1}{\sqrt{21}}\hat{i} - \frac{2}{\sqrt{21}}\hat{j} + \frac{4}{\sqrt{21}}\hat{k}$

    c. $\left(-\frac{5}{13}, \frac{12}{13}\right)$

    d. $\frac{3}{\sqrt{19}}\hat{i} + \frac{3}{\sqrt{19}}\hat{j} - \frac{1}{\sqrt{19}}\hat{k}$

## Sample Questions

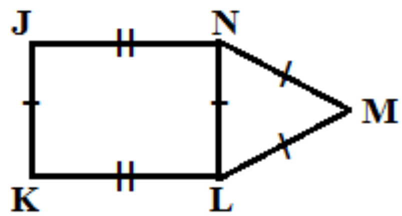
1. If $\vec{v}$ =(-2, 7) then find the magnitude of $\vec{v}$ and the angle it makes with the x-axis.

2. If $\vec{v}$ =(-3, 2, 5) then what is the vector going in the opposite direction to $\vec{v}$ that has a magnitude of 12?

3. If a wind, $\vec{v}$, is moving 15 km/h in the direction of $S60°E$ then determine $\vec{v}$ in component form. Hint: Make a diagram with a right angled triangle $w$ as the hypotenuse and use trig to solve for the length on the x-and y-axis to find the x- and y- components of $\vec{v}$

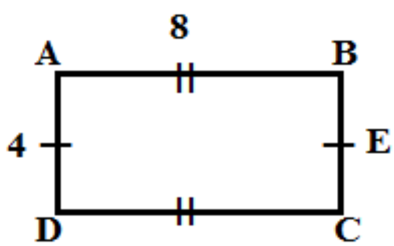
**Warm-up: Introduction to Vectors**

1. Name all equal vectors in the diagram.




2. ABCD is a rectangle with sides measuring 8 units and 4 units. E is the midpoint of BC. Find the angle between the following vectors:

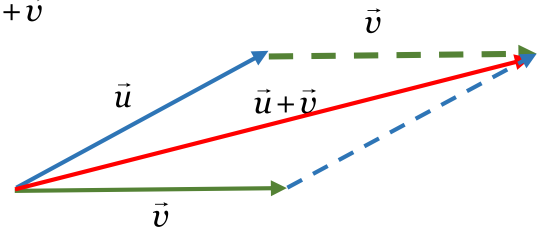
a) $\overrightarrow{DB}$ and $\overrightarrow{DC}$ b) $\overrightarrow{DE}$ and $\overrightarrow{DC}$ c) $\overrightarrow{DB}$ and $\overrightarrow{DE}$

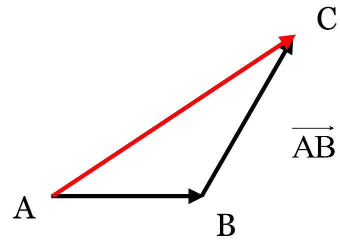



3. Draw the position vector of the point P (-2,-5). Express it in ordered pair notation, basis vector notation and column notation. Determine its magnitude and direction.

**Vector Laws:**

**Triangular Law of Vector Addition**: For vectors $\vec{u}$ and $\vec{v}$, the sum (or resultant) of $\vec{u}$ and $\vec{v}$ is a vector from the tail of $\vec{u}$ to the tip of $\vec{v}$, when the tail of $\vec{v}$ is placed at the tip of $\vec{u}$ .
Notation: $\vec{u}+\vec{v}$




$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

**Commutative Law of Addition:** $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

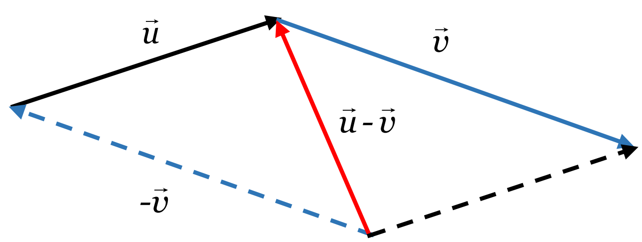
Note: sum of the vectors is the **diagonal** of the parallelogram (**Parallelogram Law of Vector Addition**)

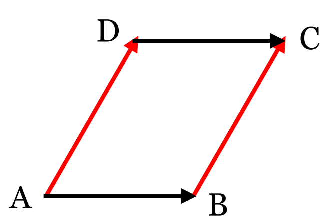
**Recap**

The **triangle law** was useful when arranging vectors tip to tail.

The **parallelogram law** was useful when arranging vectors tail to tail.

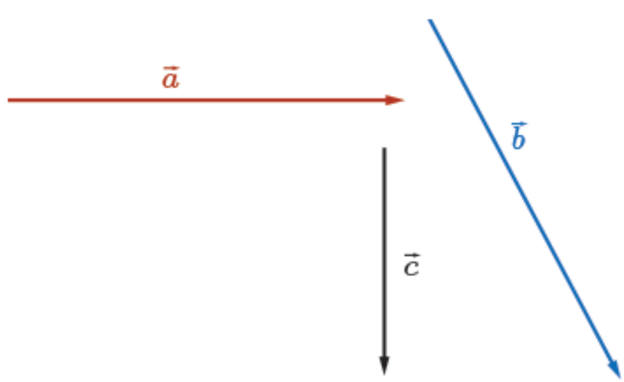
**Vector Subtraction** (adding the opposite): For vectors $\vec{u}$ and $\vec{v}$, $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$




Or $\overrightarrow{AB}-\overrightarrow{CB}=\overrightarrow{AB}+\overrightarrow{BC}$

Example 1: Determine $\vec{a}+\vec{b}-\vec{c}$.

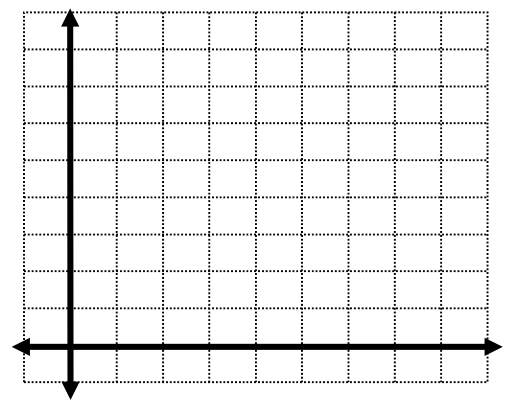


**Vector Operations** | **2-dimensions** | **3-dimensions**

- **Vector addition** $\vec{u}+\vec{v}=(u_1+v_1, u_2+v_2)$ $\vec{u}+\vec{v}=(u_1+v_1, u_2+v_2, u_3+v_3)$
- **Vector subtraction** $\vec{u}-\vec{v}=(u_1-v_1, u_2-v_2)$ $\vec{u}-\vec{v}=(u_1-v_1, u_2-v_2, u_3-v_3)$
- **Scalar multiplication:** $k\vec{u}=(ku_1, ku_2)$ $k\vec{u}=(ku_1, ku_2, ku_3)$

**Components of a vector between two points:**

The points $A(x_1, y_1)$ and $B(x_2, y_2)$ form the vector $\overrightarrow{AB}$. Using "position vectors", determine $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$. [Hint: To do this, use the triangle law of addition.]

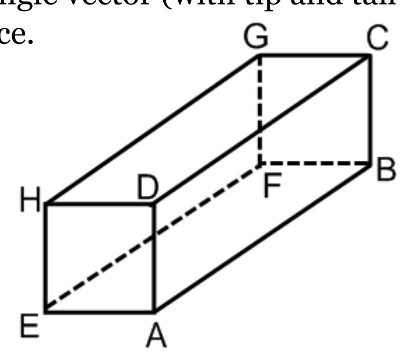


Example 2 :P(4, 5), Q(-7, 10) and R(8, -3) are three points in R².

a) Determine $\overrightarrow{QP}$ and $|\overrightarrow{QP}|$.

b) Determine $|\overrightarrow{PQ}+\overrightarrow{QR}|$.

Example 3: The diagram shows a rectangular prism. Determine a single vector (with tip and tail on the rectangular prism) that is equivalent to each sum or difference.




a) $-\overrightarrow{AE}+\overrightarrow{DA}=$

b) $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CG}=$

c) $\overrightarrow{FG}-\overrightarrow{DC}-\overrightarrow{AE}+\overrightarrow{AF}=$

Example 4: Given the vectors $\vec{u} = (2,4,-1)$, $\vec{v} = 5\hat{\imath} + 4\hat{k}$ and $\vec{w} = (-1,3,5)$, determine the following:

a) $2\vec{u} - \vec{w}$

b) $\vec{u} - \frac{\vec{v}}{2}$

c) $|\vec{u} + \vec{v} - \vec{w}|$

Example 5: A surveyor is standing at the top of a hill. Call this point the origin O. A lighthouse, L, is visible 4 km to the west and 3 km to the north. A town, T, is visible 5 km to the south and 2 km to the east. Using a vector basis in which $\hat{i}$ is a 1 km vector running east and $\hat{j}$ is 1 km vector running north, the position vectors of the lighthouse, $\overrightarrow{OL}$ and the town $\overrightarrow{OT}$ . Hence, find the vector $\overrightarrow{LT}$ and the position of the town relative to the lighthouse.

Example 6: Determine the value of $|\vec{a}+\vec{b}|$ if $|3\vec{a}| = 24\text{cm}$, $|2\vec{b}| = 10\text{cm}$ and $|3\vec{a} - 2\vec{b}| = 20\text{cm}$.

Example 7: Given that

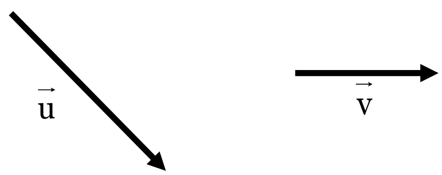
$$\vec{u} = x\vec{a} + 2y\vec{b}$$
$$\vec{v} = -2y\vec{a} + 3y\vec{b}$$
$$\vec{w} = 4\vec{a} - 2\vec{b}$$

where $\vec{a}$ and $\vec{b}$ are not collinear, find the values of $x$ and $y$ for which $2\vec{u} - \vec{v} = \vec{w}$.

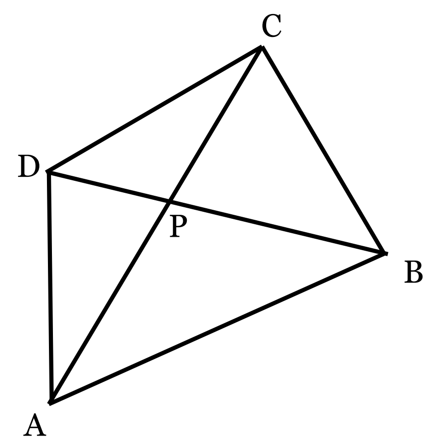
**Warm-up:**

1. For vectors $\vec{u}$ and $\vec{v}$ shown below, draw a diagram of

   a) $2\vec{u}+3\vec{v}$ b) $2\vec{u}-3\vec{v}$




2. Name a single vector equal to each combination of vectors.




   (a) $\overrightarrow{AB}+\overrightarrow{BC}$
   (b) $\overrightarrow{AB}+\overrightarrow{BD}$
   (c) $\overrightarrow{CD}+\overrightarrow{DA}$
   (d) $\overrightarrow{BC}-\overrightarrow{DC}$
   (e) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$
   (f) $\overrightarrow{DC}-\overrightarrow{BC}+\overrightarrow{BD}$
   (g) $\overrightarrow{AB}+\overrightarrow{BP}-\overrightarrow{CP}+\overrightarrow{CB}$

3. If $\vec{u}=a\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{v}=(b,-15,c)$ are collinear vectors, find

   (a) c
   (b) a relationship between a and b.

## Position Vectors

A position vector is a vector with the additional property that it is fixed at its tail to the origin O. This is not a free vector, since O is a fixed point $\overrightarrow{OP} = \vec{p}$.

Example 1: In $\Delta AOB$, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let M be the midpoint of $\overrightarrow{AB}$. Find the vector $\overrightarrow{OM}$ in terms of $\vec{a}$ and $\vec{b}$.

Example 2: OABC is a parallelogram with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. The point P lies on AB extended such that AB : BP = 2 : 1, and the point Q lies on CB such that CQ : QB = 1 : 3.

a) Express each Of these vectors terms of $\vec{a}$ and $\vec{b}$.

i) $\overrightarrow{AB}$ *ii)* $\overrightarrow{AP}$ *ii)* $\overrightarrow{OP}$ *iv)* $\overrightarrow{OQ}$

b) Hence, show that $\overrightarrow{QP} = \frac{1}{4}\vec{a} + \frac{1}{2}\vec{b}$

**Collinear points:** points that lie on the same straight line

Example 3: The position vectors of the points, A, B and C are $2\hat{i} - \hat{j} + \hat{k}$ , $3\hat{i} + 2\hat{j} - \hat{k}$ and $6\hat{i} + 11\hat{j} - 7\hat{k}$ , respectively, Show that A, B and C are collinear.

Example 4: The position vectors of a triangle ABC are $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$, and $\overrightarrow{OC} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$.

a) Find $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CA}$ and show that $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$.

b) Find $\left|\overrightarrow{OA}\right|, \left|\overrightarrow{OB}\right|$ and $\left|\overrightarrow{AB} + 2\overrightarrow{BC}\right|$

## Practice

1. Relative to a fixed origin O, the points A, B, and C have position vectors (-2, 7, 4), (-4,1,8) and (6, -5, 0) respectively.
   (a) Find the position vector of the midpoint of AB.
   (b) Find the position vector of the point D on AC such that *AD* : *DC*= 3: 1.
   Hint: Do not attempt to do this problem with distances. How would you split a line into 4 equal segments using midpoints?

2. Relative to a fixed origin o, the point A, B, and C have position vectors (6, -2, -4), (12, -7, -4), and (6, 1, -8) respectively.
   (a) Find the position vector of the point M, the midpoint of BC.
   (b) Show that O, A, and M are collinear points.

3. Given that $\vec{p}$ = (1, -2, 4), $\vec{q}$ = (-1, 2, 2), and $\vec{r}$ = (2, -4, -7). Find the value of *t* such that $\vec{p}+t\vec{q}$ is parallel to $\vec{r}$ .

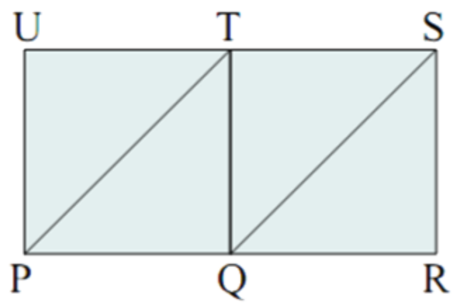
4. The diagram contains two squares. Express each difference as a single vector

   a) $\overrightarrow{SQ}-\overrightarrow{ST}$

   b) $\overrightarrow{QT}-\overrightarrow{QP}$

   c) $\overrightarrow{PR}-\overrightarrow{QS}$

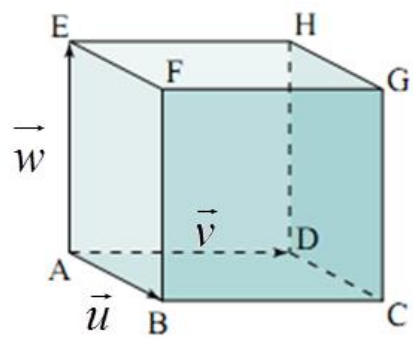
   d) $\overrightarrow{PT}-\overrightarrow{TS}$



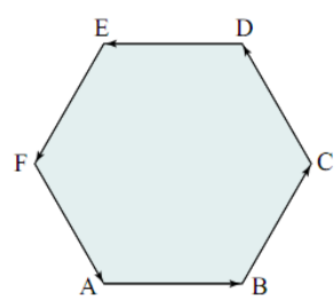

5. The diagram shows a cube, where
   $\overrightarrow{AB}=\vec{u}$ ,$\overrightarrow{AD}=\vec{v}$ and $\overrightarrow{AD}=\vec{v}$ .Determine a single vector equivalent to each of the following.

   a) $\vec{u}-\vec{v}+\vec{w}$

   b) $\vec{u}-\vec{v}-\vec{w}$




6. The diagram shows a regular hexagon.
   Prove that: $\overrightarrow{AB}-\overrightarrow{BC}+\overrightarrow{CD}-\overrightarrow{DE}+\overrightarrow{EF}-\overrightarrow{FA}=\vec{0}$



## 3-5 Warm Up

1. Determine the values of $r$ and $s$ given that $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix}$ is parallel to $\vec{b} = \begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$.

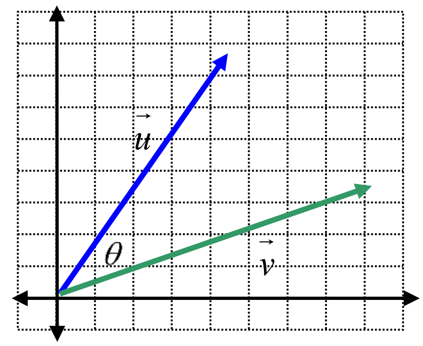
2. Quadrilateral ORST has position vectors $\vec{r}$, $\vec{s}$ and $\vec{t}$. Point A is the midpoint of RS and point B divides ST such that SB:BT=2:5. Express each of these vectors in terms of $\vec{r}$, $\vec{s}$ and $\vec{t}$.

   a) $\overrightarrow{RS}$ b) $\overrightarrow{ST}$ c) $\overrightarrow{OB}$ d) $\overrightarrow{AB}$

## Dot Product of 2 Vectors – aka Scalar (Inner) Product

*Dot Product* :
- defined as: **(horizontal displacement of an object)**
- dot product involves two scalars
- result is a scalar ie) positive/negative/zero

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)}$$




**Note: Vectors need to be tail to tail**

| **angle value** | $\cos(\theta)$ **value** | $\vec{u} \cdot \vec{v}$ |
|---|---|---|
| $0^\circ \leq \theta \leq 90^\circ$ | | |
| $\theta = 90^\circ$ | | |
| $90^\circ \leq \theta \leq 180^\circ$ | | |

Example 1: Given vectors $\vec{u}$ and $\vec{v}$, where $|\vec{u}|=10$ and $|\vec{v}|=13$ and the angle between them is $150^\circ$, calculate $\vec{u} \cdot \vec{v}$.

## Dot Product Properties

1) ***Commutative***: $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

2) ***Distributive** over vector addition:* $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

3) ***Associative** over scalar multiplication:* $m(\vec{u} \bullet \vec{v}) = (m\vec{u}) \bullet \vec{v} = \vec{u} \bullet (m\vec{v})$

$(m\vec{u}) \bullet (n\vec{v}) = mn(\vec{u} \bullet \vec{v})$

Example 2:Evaluate $\hat{j} \cdot \hat{j}$ and $\hat{i} \cdot \hat{j}$.

Example 3: What is the dot product of a vector $\vec{u}$ with itself? ($\theta = 0^\circ$).

Example 4: If vectors $3\vec{p}+\vec{q}$ and $\vec{p}-3\vec{q}$ are perpendicular and $|\vec{p}| = 2|\vec{q}|$, determine the angle between the non-zero vectors $\vec{p}, \vec{q}$.

**How to Evaluate Dot Product of Algebraic Vectors**

**2-dimensions**

$\vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2)$

$\vec{u} \bullet \vec{v} = u_1v_1 + u_2v_2$

**3-dimensions**

$\vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, u_3)$

$\vec{u} \bullet \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

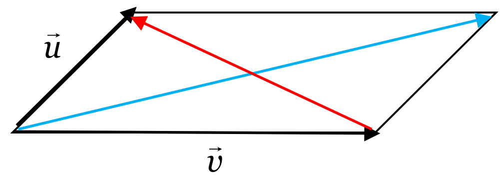
Example 5. Given that vectors $\vec{u} = \begin{pmatrix} k+2 \\ 5 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} k+1 \\ -6 \end{pmatrix}$ are perpendicular, solve for *k*.

**Using dot product to find angle between two vectors**

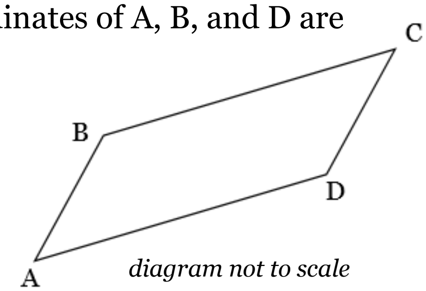
The formula $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ can be rearranged to make solving for $\theta$ simpler.

$$\boxed{\cos(\theta) = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|}}$$

Example 6: A parallelogram is bounded by vectors $\vec{u} = (1,2)$ and $\vec{v} = (3,-2)$. Find the angle between the diagonals of the parallelogram




Example 7: The diagram shows a parallelogram ABCD. The coordinates of A, B, and D are A(1,2,3) ,B(6,4,4) and D(2,5,5).




(a) Show that $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$, find $\overrightarrow{AD}$ and hence show that $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$.

(b) Find the coordinates of point C.

(c) Find $\overrightarrow{AB} \bullet \overrightarrow{AD}$ and hence find angle A.

**Practice**

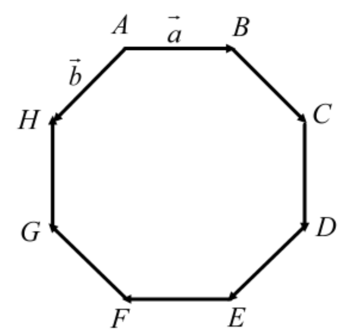
1. Given that $|\vec{a}| = 2, |\vec{b}| = 3$, and $\theta = 120^\circ$ expand and simplify $\left(3\vec{a}+4\vec{b}\right)\bullet\left(5\vec{a}+6\vec{b}\right)$.
2. The points A(-1,1), B(2,0), and C(1,-3) are vertices of a triangle.
   (a) Show that this triangle is a right triangle.
   (b) Calculate the area of triangle ABC.
   (c) Calculate the perimeter of triangle ABC.
   (d) Calculate the coordinates of the fourth vertex D that completes the rectangle of which A, B, and C are the other three vertices

3. If $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$, prove that the non-zero vectors $\vec{a}, \vec{b}$ are perpendicular. What could this look like?

4. Given the vectors $\vec{u} = [1,0,1]$, $\vec{v} = 2\hat{\imath} + \boldsymbol{m}\hat{\jmath} + 2\hat{k}$ find the value(s) of $\boldsymbol{m}$ if the angle between $\vec{u}$ and $\vec{v}$ is $45^\circ$.
5. Find the angle between the given vector and the axis.

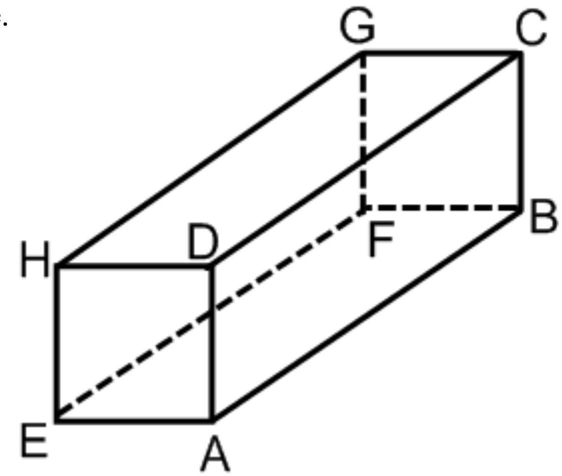
   a) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ and negative x-axis

   b) $\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$ and positive x-axis
6. ABCDEFGH is a regular octagon with sides of unit length. (Recall interior angles are 135°). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AH} = \vec{b}$. Prove that $\overrightarrow{BC} = \vec{b} + \sqrt{2}\,\vec{a}$.



# Mid-Review: Geometric Vector

1. The diagram shows a parallelepiped. Determine a single vector (with head and tail on the parallelepiped) that is equivalent to each sum or difference.

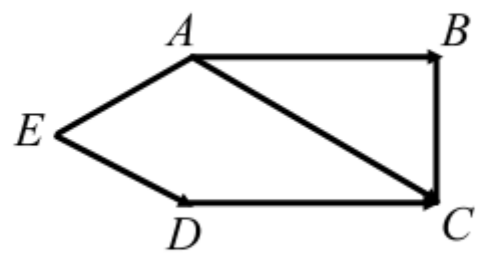



$a)\quad -\overrightarrow{AE}+\overrightarrow{DA}=$ ________

$b)\quad \overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CG}=$ ________

$c)\quad \overrightarrow{FG}-\overrightarrow{DC}-\overrightarrow{AE}+\overrightarrow{AF}=$ ______

2. ABCDE is a pentagon such that $\overrightarrow{AB}=\overrightarrow{DC}$ and $\overrightarrow{AC}=2\overrightarrow{ED}$ .write each vector in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ .




a) $\overrightarrow{EC}=$

b) $\overrightarrow{CB}=$

c) $\overrightarrow{AE}=$

3. If $\vec{a}$ and $\vec{b}$ are unit vectors that make an angle of $60^0$ with each other, calculate $\left|3\vec{a}-4\vec{b}\right|$.

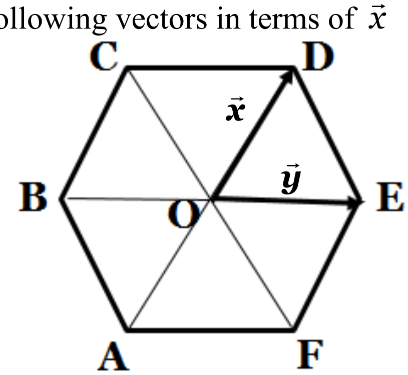
4. If $\frac{2}{3}\vec{x}=\vec{a}+\frac{1}{3}\vec{b}$, $2\vec{y}=-3\vec{a}+\vec{b}$ express $2\vec{a}-5\vec{b}$ in terms of $\vec{x},\vec{y}$.

5. Given that

$$\vec{u}=x\vec{a}+2y\vec{b}$$
$$\vec{v}=-2y\vec{a}+3y\vec{b}$$
$$\vec{w}=4\vec{a}-2\vec{b}$$

where $\vec{a}$ and $\vec{b}$ are not collinear, find the values of $x$ and $y$ for which $2\vec{u}-\vec{v}=\vec{w}$.

6. Using the regular hexagon ABCDEF shown, express each of the following vectors in terms of $\vec{x}$ and $\vec{y}$.




a) $\overrightarrow{DA}=$

b) $\overrightarrow{DE}=$

c) $\overrightarrow{BF}=$

# Mid-Review: Geometric Vector

7. Given $\vec{p} = [2,-3], \vec{q} = [-1,4]$, evaluate $|3\vec{p} - 2\vec{q}|$.

8. Given the point $P(4,-3)$ where $\overrightarrow{PQ} = [7,-4]$ find

   a) coordinates of Q  b) $|\overrightarrow{PQ}|$ c) a unit vector in the direction of $\overrightarrow{QP}$

9. If $\vec{u}$ =[1,4,-2], $\vec{v} = -2\hat{i} - 3\hat{j}$ and $\vec{w}$ =[-1,-3,1], find:

   a) $|3\vec{v} + 3\hat{i} - 2k|$

   b) a unit vector with the same direction as $\vec{u}$.

   c) Find the angle between $\vec{v}$ and $\vec{w}$

10. The points A(-1,2,-1) ,B(2,-1,3), and D(-3,1,-3) are three vertices of parallelogram ABCD. Find the coordinate of C.

11. Vectors [2, -a, 1] and [−2, 2, -a+1] are collinear. Find the value of a.

12. The vectors $\vec{u}$ and $\vec{v}$ have lengths 2 and 1 respectively. The vectors $\vec{u} + 5\vec{v}$ and $2\vec{u} - 3\vec{v}$ are perpendicular. Determine the angle between $\vec{u}$ and $\vec{v}$.

## 3-6 Warm Up

1. Find the dot (scalar) product of $\vec{u}$ and $\vec{v}$ for the following:
   a) $|\vec{u}| = 5, |\vec{v}| = 8, \ \theta = 13°$
   b) $\vec{u} = (5, 6), \overrightarrow{v} = (-2, 3)$

2. Find the angle between the vectors:

   a) $\vec{u} = (-6, 1)$ and $\overrightarrow{v} = (-5, 3)$

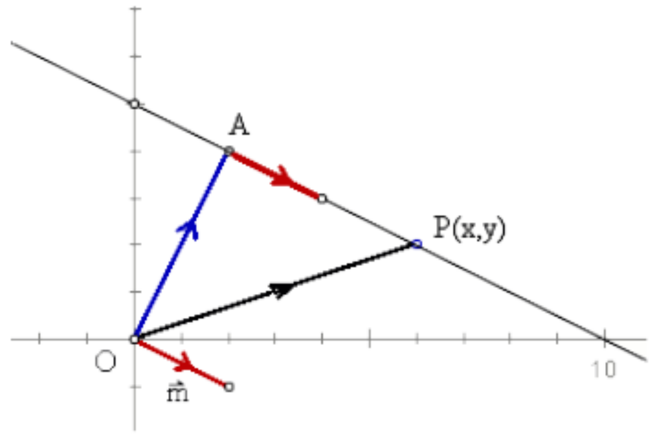
   b) $3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $4\hat{\imath} - 5\hat{\jmath} - 3\hat{k}$

3. Find the value of c so that the vectors $\hat{\imath} + \hat{\jmath} + \hat{k}$ and $c^2\hat{\imath} - 2c\hat{\jmath} + \hat{k}$ are perpendicular.

## Equations of Lines in R²

In R² vectors can be used to define a line .Two new forms of the equation of the line are the **Vector Equation of a Line** and **Parametric Form of the Equation of a Line**. We start by defining the former.

Vectors can be used to locate points on a line as shown in the diagram at right. If A is a given point on the line and $\vec{m}$ is a vector parallel to the line, $\overrightarrow{OP} = \overrightarrow{OA} + t\vec{m}$ can be used to locate any point P(x, y) on the line.




This equation is called the **Vector Equation** of the line.

$$\overrightarrow{OP} = \overrightarrow{OA} + t\vec{m}$$

$\overrightarrow{OA}$ is called a **Position Vector**

$\vec{m}$ is called a **Direction Vector**

t is called a **Parameter** (any real number)

### Vector Equation of Lines in R²

Another way to write this equation using variables is $\vec{r} = \vec{r}_o + t\vec{m}$ . By substituting $\vec{r} = (x, y)$, $\vec{r}_o = (x_o, y_o)$ and $\vec{m} = (m_1, m_2)$ into this equation we get another form of the vector equation.

**The Vector Equation of a Line in R²**

$\vec{r} = \vec{r}_0 + t\vec{m}$ OR $(x, y) = (x_0, y_0) + t(m_1, m_2)$

where

- $t \in R$ is a parameter
- $\vec{r} = (x, y)$ is a position vector to any unknown point on the line
- $\vec{r}_0 = (x_0, y_0)$ is a position vector to any known point on the line
- $\vec{m} = (m_1, m_2)$ is a direction vector parallel to the line

Example 1:

a) Write a vector equation of a line passing through the points A(1, 4) and B(3, 1).

b) Determine two more points on the line.

c) Determine if the point (2, 3) is on this line.

NOTE: Vector equations are **NOT** unique!

The vector equation can be separated into two parts, one for each variable. These are called **parametric equations** of a line.

| **The Parametric form of the Equation of a Line in $R^2$** |
|---|
| For a line with equation $(x, y) = (x_0, y_0) + t(m_1, m_2)$, the parametric equations are<br>$x = x_0 + tm_1$<br>$y = y_0 + tm_2$ where $t \in R$ (the parameter) |

Example 2: Rewrite your vector equation from Example 1(a) in parametric form.

NOTE: Again, like vector equations, parametric equations are not unique as we can use the coordinates of any point on the line and any scalar multiple of the direction vector.

Example 3: A line $L_1$ is defined by $x = 3 + t$ and $y = -5 + 2t$.

a) Find the coordinates of two points on this line.

b) Find the y-intercept of the line.

c) Write the vector equation for $L_1$.

d) Determine if $L_1$ is parallel to $L_2$: $x = 1 + 2t$, $y = -9 + 4t$.

**Symmetric Equation** of a line.

| **The Symmetric form of the Equation of a Line in $R^2$** |
|---|
| For a line with equation $(x, y) = (x_0, y_0) + t(m_1, m_2)$, the symmetric equation is |
| $\frac{x - x_0}{m_1} = \frac{y - y_0}{m_2}, m_1, m_2 \neq 0$ |

**What happens if $m_1$ or $m_2$ is zero?**

Let suppose $m_2=0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x for t. We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x - x_0}{m_1}, \; y = y_0$$

Example 4: Write all three forms of the equation of the line that passes through the points

A (2,-1) and B(4,-1).

Example 5: Consider the line with Cartesian equation $4x + 5y + 20 = 0$.

a) Determine its slope. How does the slope compare to the Cartesian equation?

b) Determine a vector equation of this line. How does the direction vector relate to the slope?

c) Determine a position vector that is perpendicular to the line (e.g. a normal vector). How does the normal vector compare to the Cartesian equation?

**NOTE: For a line with equation $\mathbf{Ax+By+C=0}$,**

- the slope of the line is ______ and a direction vector $\vec{m}$ =__________.
- the normal vector is $\vec{n}=$ _________.

**You Try!**

Determine equivalent vector, parametric, symmetric and Cartesian equations of the line $y=\frac{3}{4}x+2$.

## Equations of Lines in R3

As in R2, a direction vector and a position vector to a known point on a line are all that are needed to define a line in R3**.**

**The Vector Equation of a Line in R3**

$\vec{r} = \vec{r}_0 + t\vec{m}$ OR $(x, y, z) = (x_0, y_0, z_0) + t(m_1, m_2, m_3)$

where

- $t \in$ R is a parameter
- $\vec{r} = (x, y, z)$ is a position vector to any unknown point on the line
- $\vec{r}_0 = (x_0, y_0, z_0)$ is a position vector to any known point on the line
- $\vec{m} = (m_1, m_2, m_3)$ a direction vector parallel to the line

**The Parametric form of the Equation of a Line in R3**

For a line with equation $(x, y, z) = (x_0, y_0, z_0) + t(m_1, m_2, m_3)$, the parametric equations are

____________________

____________________

____________________ where $t \in$ R

Overall, the various new forms of lines in ***R*2** can be extended to lines in ***R*3.**

**<u>Comparison of equations of a line in *R*2 and *R*3</u>**

| | Equation of a line in ***R*2** | Equation of a line in ***R*3** |
|---|---|---|
| **Scalar** | $Ax + By + C = 0$ | |
| **Vector** | $(x, y) = (x_o, y_o) + t(m_1, m_2)$ | |
| **Parametric** | $x = x_o + tm_1$<br>$y = y_o + tm_2$ , $t \in \mathbb{R}$ | |
| **Symmetric** | $\frac{x - x_o}{m_1} = \frac{y - y_o}{m_2}$<br>where $m_1, m_2 \neq 0$ | |

Example 6: A line passes through points A(2, -2, 5) and B(0, 6, -5).

a) Write a vector equation for the line.

b) Write parametric equations for the line.

c) Write symmetric equations for the line.

d) Determine if the point C(0, -10, 9) lies on the line.

**Thinking Question: Why can't a normal vector and a point define a line in $R^3$?**

## Practice

1. Determine if the following points are on the line $\ell$:[−4,3]+t[3,2].
a) (−1,5) b) (−16,−5)
2. For the line defined by $\ell:\begin{cases} x=-3-t \\ y=2+2t \end{cases}$, state the coordinates of
a) the y-intercept
b) the x-intercept
c) the point where x=12
d) the point where y=38

3. Rewrite each of the equations below into the specified form.
a) $\ell$:[7,2]+t[3,−2] into parametric form
b) $\ell:\begin{cases} x=32-3t \\ y=26+4t \end{cases}$ into vector form
4. Find the equation of the line and write in the specified form:
a) the line parallel to $\vec{m}=[2,3]$ that hits the point (1,4), in parametric form.
b) the line that passes through the points (2,4) and (5,13), in vector form.
c) the vertical line through (4,−2), in parametric form.
d) the line with the same x-intercept as $\ell_1$:[3,6]+t[1,−2], and the same y-intercept as $\ell_2$:[8,4]+s[−1,3], in vector form.

5. Given the line $\ell$:[7,3,1]+t[−1,3,1], determine if the following lines are parallel, perpendicular, or coincident to it.
a) $\ell_2$:[2,−3,4]+t[5,1,2]
b) $\ell_3$: x=1+t , y=21−3t , z=7−t
c) $\ell_4$:[5,3,2]+t[−2,6,2]
d) $\ell_5$:[3,7,−2]+t[4,6,1]

6. If the points (4,2,7),(6,19,−4), and (80,b,c) lie on the same straight line, find the values of b and c.

7. Determine the angle between each pair of lines:
a) $\ell_1$:[4,5,−2]+t[3,−1,−1] $\ell_2$:[4,5,−2)+s[−2,−3,2]
b) $\ell_1: \frac{x-5}{3}=\frac{y+2}{5}=z-2$ $\ell_2: \frac{x-5}{8}=y+2=\frac{2-z}{3}$
8. Find, in parametric form, the equation of a line perpendicular to both $\ell_1$:[3,7,−2]+t[3,−1,−1] and $\ell_2$:[8,−3,−3]+t[−2,−3,2] that passes through (5,0,0).

9. Find, if possible, the value(s) of k such that the lines $\ell_1$:[9,3,2]+t[3,k,−15] and $\ell_2$:[−5,4,−2]+t[10,12,50] are:
a) parallel b) perpendicular

10. Point $P_1$ lies on the line $\ell_1$:[4,4,−3]+t[2,1,−1],t∈ℝ, and point $P_2$ lies on the line $\ell_2$:[−2,−7,2]+s[3,2,−3]. If the vector $\overrightarrow{P_1P_2}$ is perpendicular to both $\ell_1$ and $\ell_2$, determine the coordinates of $P_1$ and $P_2$.

## 3-7 Warm Up

1. Determine whether $l_1: \begin{matrix} x = 2 - t \\ y = 5t \end{matrix}$ and $l_2: x - 3 = \frac{1-y}{5}$ are coincident.

2. Develop vector, parametric, symmetric and Cartesian equations of the line through the point (3, -5) and is perpendicular to the line: $\begin{matrix} x = 3t - 5 \\ y = 2 + t \end{matrix}$.

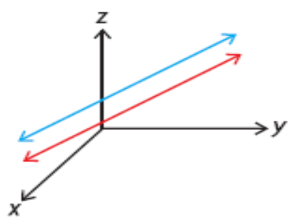
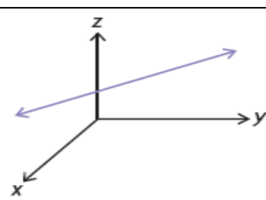
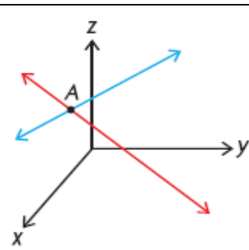
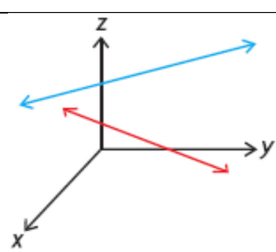
3. Determine the Cartesian equation of the line passing through the point P(5, 4) and perpendicular to $\vec{u} = \binom{2}{-6}$.

## The Intersection of Two Lines in $R^2$ and $R^3$

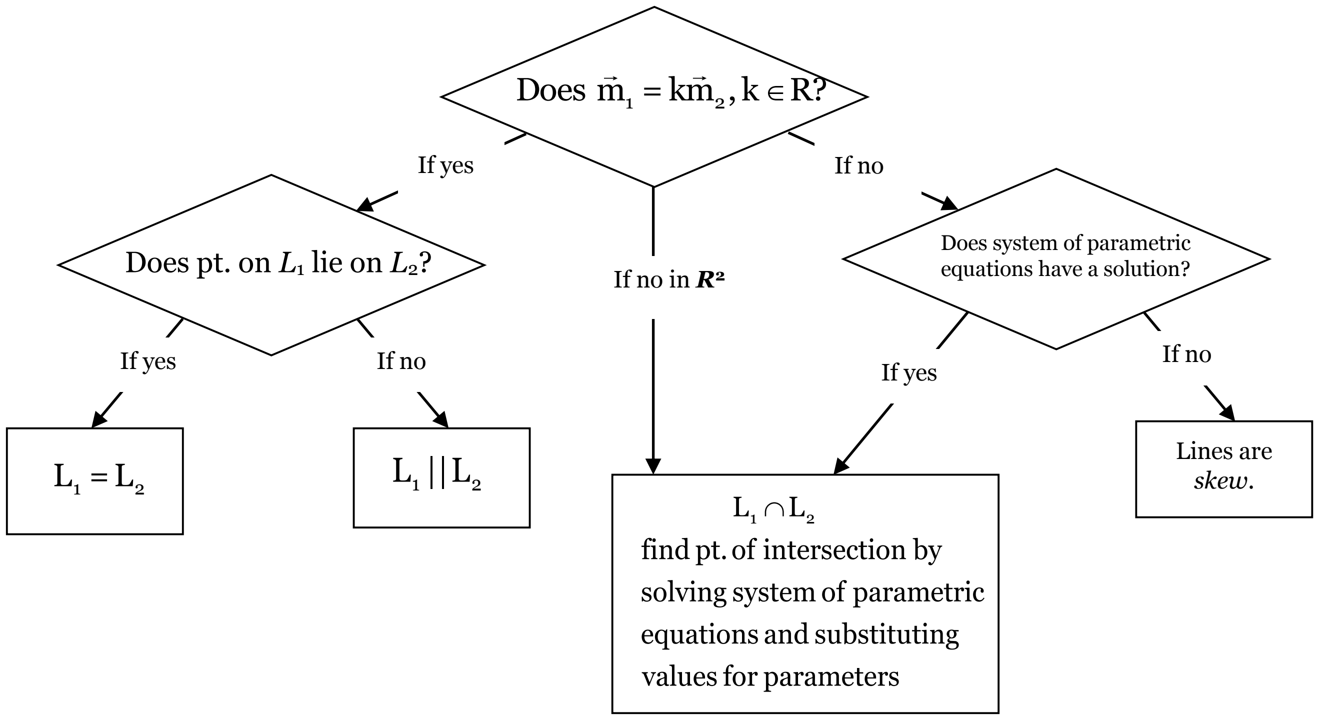
## In $R^2$

| Lines may… | Diagram | Number of solutions |
|---|---|---|
| be parallel | | |
| coincide (be coincident) | | |
| intersect | | |

In $\boldsymbol{R^3}$ (the **Notation** is the same as for $\boldsymbol{R^2}$)

| Lines may… | Conditions | Number of solutions |
|---|---|---|
| be parallel |  | |
| coincide (be coincident) |  | |
| intersect<br>(and are therefore<br>coplanar) |  | |
| ***skew***<br>(do not intersect **and** are<br>not parallel) |  | |

## Method for Determining Line Situation




## Examples for $R^2$:

Are each of the following pairs of lines parallel, coincident or intersecting? If the lines intersect, find the point of intersection.

1) $L_1 : 2x + 3y - 30 = 0$
$L_2 : x - 2y + 13 = 0$

2) $L_1 : (x, y) = (18, -2) + t(3, -2)$

$L_2 : (x, y) = (-5, 4) + s(2, 1)$

3) $L_1 : (x, y) = (1, 3) + t(4, 2)$

$L_2 : \dfrac{x - 2}{2} = y - 1$

**Examples for $R^3$:**

Are each of the following pairs of lines parallel, coincident, intersecting or *skew* ? If the lines intersect, find the point of intersection.

4) $L_1 : (x, y, z) = (-1, 1, 0) + t(3, 4, -2)$
$L_2 : (x, y, z) = (-1, 0, -7) + s(2, 3, 1)$

5) $L_1 : (x,y,z) = (2,1,0) + t(1,-1,1)$

$L_2 : (x,y,z) = (3,0,-1) + s(2,3,-1)$

6) $L_1 : x = 1 - 2s, y = s, z = -1 - s, s \in R$

$L_2 : \dfrac{x+1}{-2} = \dfrac{1-y}{-1} = z - 2$

**Practice**

1. Find the value(s) of $a$ and $b$ that make the lines $\begin{array}{l}\ell_1 : \vec{r} = [3,0,-2] + t[3,1,-3] \\ \ell_2 : \vec{r} = [15,4,a] + s[5,b,-5]\end{array}$

   a) Coincident
   b) Parallel and distinct
   c) Intersecting
   d) Skew

2. Determine the parametric equations of a line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x-4}{3} = \frac{y+1}{5} = \frac{z-4}{2}$ and $\frac{x}{6} = \frac{y-7}{10} = \frac{z+3}{5}$ and passes through the point $(5,0,-2)$.

3. Find the vector equation of the line through the point (8,10,10) that meets the line $\frac{x+8}{-1} = \frac{y-11}{3} = \frac{z-1}{4}$ at 90° angles.

4. Lines $\ell_1 :\ \vec{r} = [2,1,3] + t[6,-4,-1]; t \in R$ and $\ell_2 : \begin{cases} x-1 = -s \\ y-6 = \mathbf{a}s \\ z-2 = \mathbf{b}s \end{cases}$ are intersecting at point (-1,3,2). What are the possible values of **a** and **b**?

5. Find all values of ***k*** for which the following lines **do not** intersect.

$$l_1 : \begin{cases} x = -1 + 2r \\ y = 3k + r \\ z = 1 + 3r \end{cases} \quad \text{and} \quad l_2 : \vec{r} = [1,0,-2] + t[-2,3,1]$$

6. Determine the point(s) of intersection between line $x - 2 = t, y + 2 = 3t, 2 + z = 2t, t \in \mathbb{R}$ and sphere with equation $x^2 + y^2 + z^2 = 100$.

7. Determine if the following lines are parallel, skew or intersecting. In the case the lines are intersecting, find the point of intersection.
   $L_1$: [x, y, z] = [-3, 1, 4] + t[1,-1,-4] and $L_2$: [x, y, z) = [1,4,6] + s[6,1,7]

8. Determine why the lines $\vec{r} = [1,3,4] + s[2,3,5]$ and $\vec{v}$ = [1,1,1]+ t[2, 2,-2]are not perpendicular.

## 3-8 Warm Up

1. Prove that the lines $l_1: (x-1, y-4, z) = s(-4, 2, 6), s \in \mathbb{R}$ and $l_2: (x, y, z) = (-3, 3, 0) + t(0, 1, 2),\ t \in \mathbb{R}$ lie in the same plane.

1. Write each of the following lines in scalar, vector, parametric, and symmetric form.

| Scalar (Cartesian) | Vector | Parametric | Symmetric |
|---|---|---|---|
| $2x + 3y - 6 = 0$ | | | |
| | | | $\frac{x-2}{2} = \frac{4-2y}{-3}$ |
| | $\vec{r} = [2,-3,1] + t[1,-1,4]$ | | |
| | | $\begin{cases} x = -2t \\ y = -1 + 3t \\ z = 3t + 2 \end{cases}$ | |

2. Determine the **exact** value(s) of k that would make the following lines intersect at $60^\circ$ angle.

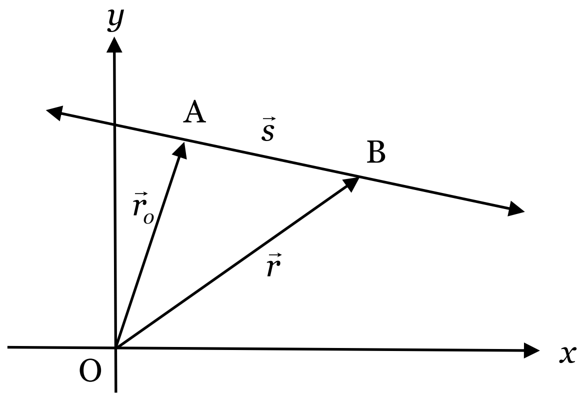
$L_1 : \vec{r} = [17,-16] + t[1,k]$ and $L_2 : \frac{x-25}{1} = \frac{y-3}{-1}$

$$\cos(60^\circ) = \frac{\vec{m_1} \bullet \vec{m_2}}{|\vec{m_1}||\vec{m_2}|}$$

3. Determine parametric equations of a line that is parallel to $\vec{r}_1 = \left[3, \frac{5}{2}, -5\right] + t[-1, 1, 0]$ and passes through the x-intercept of $\begin{cases} x = 2 + 3t \\ y = \frac{1}{2} + \frac{1}{4}t \\ z = 6 + 3t \end{cases}$.

## Application to Constant Motion in Two-Dimensional Space

Consider an object moving with constant velocity , $\vec{v}$, modeled as a point particle moving along an arbitrary path in the *xy*-plane. We assume that we are able to detect the particle's position at any point and to measure the corresponding clock time. Two positions *A* and *B* in the particle's path are shown. Let the vectors that locate these positions be $\vec{r}_0$ and $\vec{r}$ , respectively. The displacement , $\vec{s}$ , can be expresses as $\vec{s} = \vec{r} - \vec{r}_o$ .




Therefore, $displacement = \vec{s} = \overrightarrow{AB}$

$$= \vec{r} - \vec{r}_o$$

Since $\vec{s} = t\vec{v}$ , *where* $\vec{v}$ *is the velocity*

$$\vec{r} - \vec{r}_o = t\vec{v}$$

or $\boxed{\vec{r} = \vec{r}_o + t\vec{v}}$

Suppose that $\vec{r}_o = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ km and the cyclist's velocity is $\begin{pmatrix} 10 \\ -2 \end{pmatrix}$ km/h. B is an arbitrary position on the cyclist's path, so $\overrightarrow{OB} = \vec{r}$ can be written as $\begin{pmatrix} x \\ y \end{pmatrix}$ so that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 10 \\ -2 \end{pmatrix}$ for any value of $t \geq 0$. The position vector of the cyclist can be found at any time by replacing *t* with the appropriate numerical value.

Example 1: An object, P, moves in a straight line with constant velocity. Its position vector, relative to an origin, O, is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}, t \geq 0$, where *t* is measured in hours and displacement is measured in kilometers

(a) Find the coordinates of P (i) initially (ii) after 5 hours.

Another object, Q, also moves in a straight line with constant velocity so that its position vector at time, t, is given $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \end{pmatrix}, t \geq 0$

(b) Show that the paths of P and Q intersect, but P and Q do not collide.

Example 2. A cyclist is traveling at a speed of 26 km/h in a direction $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$ relative to an origin, O

She starts at point A(-2, 10) and, after one hour she has reached point B

(a) Write down a unit vector parallel to the cyclist's velocity and use it to find her velocity as a column vector

(b) Find (i) $\overrightarrow{OA}$ (ii) $\overrightarrow{AB}$ (iii) $\overrightarrow{OB}$

(c) Find the coordinates of B

After t hours, she is at point P.

(d) On Cartesian axes, show the cyclist's path and the points O, A B and an arbitrary point P.

(e) Find (i) $\overrightarrow{AP}$ (ii) $\overrightarrow{OP}$ (iii) $\overrightarrow{OP} \bullet \overrightarrow{AP}$ in terms of $t$

(f) Hence, find the time, to the nearest minute, for the cyclist to be closest to O and the distance $|\overrightarrow{OP}|$ at this time.

## Practice

1. The position of two submarines $S_1$ and $S_2$ at time t hours are given by the formulas
   $S_1$: $[x, y, z] = [2, 1, -4] + t\,[2, 1, 2]$
   $S_2$ : $[x, y, z] = [1, 1, 1] + t\,[2, 0.5, -4]$

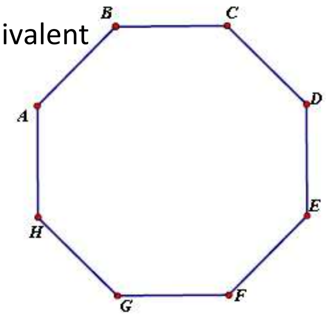
   (a) What is the speed of the first submarine?
   (b) Determine if the paths of the two submarines will intersect.
   (c) Determine if the two submarines will collide.

2. Position in km of a helicopter is given by $\mathbf{r} = \begin{pmatrix} 17 \\ -11 \end{pmatrix} + t\begin{pmatrix} -20 \\ 21 \end{pmatrix}$ where 't' is the number of hours after 8:00 a.m. Sherwood Park is at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find:
   (a) distance from Sherwood Park at 10:00 a.m.
   (b) time when the plane is 123 km west and 133 km north of Sherwood Park

3. A particle is moving with a constant velocity along line L. Its initial position is A(6, –2, 10) and after one second it has moved to B(9, –6, 15).
   (a) Find the velocity vector $\overrightarrow{AB}$ and find the speed of the particle.
   (b) Write down a possible vector equation of the line L.

4. The position of ship A is given by $\mathbf{a} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and ship B by $\mathbf{b} = \begin{pmatrix} -13 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ where the distance is in kilometres and t is the number of hours after 9:00 A.M. The base is at the origin. Find:
   (a) the position of ship A at 1:00 p.m. relative to the base.
   (b) distance between the 2 ships at 1:00 p.m.
   (c) the time when they would collide.
   (d) the speed of ship A.

# Unit 2 – Algebraic Vectors Review




1. The diagram on the right shows a regular octagon. Write a single vector that is equivalent to each vector expression
   a. $\overrightarrow{HA} + \overrightarrow{AB}$ $[\overrightarrow{HB}]$
   b. $\overrightarrow{GH} - \overrightarrow{GF}$ $[\overrightarrow{FH}]$
   c. $\overrightarrow{FE} + \overrightarrow{BA}$ $[\vec{0}]$
   d. $\overrightarrow{GA} - \overrightarrow{EH} + \overrightarrow{DG}$ $[\vec{0}]$

2. Given $\vec{u} = [-2, y]$ and $\vec{u}$ makes a 120° with the x-axis. Determine the value of y and $|\vec{u}|$.
   $[y = 2\sqrt{3},\ |\vec{u}| = 4]$

3. Using vectors show that the three points $A(2, -3, 7)$, $B(7, 12, -3)$, and $C(-2, -15, 15)$ are collinear.

4. Determine the value of k so that $\vec{u} = [k, 3]$ and $\vec{v} = [k, 2k]$ are perpendicular. $[k = -6]$

5. Solve for x if $\vec{u} = [3x, 7], \vec{v} = [5x, x], and\ |\vec{u} + \vec{v}| = 10x.$ $\left[x = \frac{7}{5}\right]$

6. Given that $\vec{u} = 3\vec{x} - \vec{y}$ and $\vec{v} = 2\vec{x} + 5\vec{y}$
   a. Express $\vec{w} = \vec{u} + \vec{v} - 2\vec{x} + \vec{y}$ in terms of $\vec{x}$ and $\vec{y}$. $[\vec{w} = 3\vec{x} + 5\vec{y}]$
   b. If $\vec{x} = [1, 3]$ and $\vec{y} = [-2, 5]$ then
      i. determine $|\vec{w}|$. $[\sqrt{1205}]$
      ii. determine the angle $\vec{w}$ makes with the x-axis. $[101.6°]$

7. Given that $|\vec{a}| = 10$, $|\vec{b}| = 15$, and $|\vec{a} - \vec{b}| = 11$
   a. find the angle between $\vec{a}$ and $\vec{b}$. $[47.156°]$
   b. calculate $|\vec{a} + \vec{b}|$. $[23]$

8. In a quadrilateral ABCD, T is the midpoint of the side AB. U is the midpoint of the side CD. L is the midpoint of the diagonal AC, and M is the midpoint of the diagonal BD. Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$ and $\overrightarrow{CD} = \vec{c}$.
   a. Show that $\overrightarrow{AD} + \overrightarrow{BC} = 2\overrightarrow{TU}$.
   b. Show that $\overrightarrow{AD} + \overrightarrow{CB} = 2\overrightarrow{LM}$.

9. a. If $\cos(\theta) = X$ and $\theta$ is obtuse, what do you know about the sign of X?
   b. If the angle between $\vec{u}$ and $\vec{v}$ is obtuse, what do you know about the value of $\vec{u} \cdot \vec{v}$?
   c. Determine the value(s) of k so that the angle between $\vec{x} = [11, 3, 2k]$ and $\vec{y} = [k, 4, k]$ is obtuse.
   $\left[-4 < k < -\frac{3}{2}\right]$

10. If $\vec{u} = [1, 4, -2], \vec{v} = -2\hat{\imath} - 3\hat{\jmath}, and\ \vec{w} = [-1, -3, 1]$ find
    a. $|3\vec{v} + 3\hat{\imath} - 2\hat{k}|$. $[\sqrt{94}]$
    b. $\hat{u}$. $\left[\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}\right]$
    c. the angle between $\vec{v}\ and\ \vec{w}$. $[23.09°]$

d. a vector with a magnitude of 7 in the opposite direction of $\vec{w}$. $\left[\frac{7}{\sqrt{11}}[1,3,-1]\right]$

11. The points $A(-1,2,-1)$, $B(2,-1,3)$ and $D(-3,1,-3)$ are three vertices of parallelogram ABCD.
    a. Find the coordinate of C . $[C(0,-2,1)]$
    b. Verify that the vector $-10\hat{\imath}+2\hat{\jmath}+9\hat{k}$ is perpendicular to $\overrightarrow{AB}$.

12. If the points $A(1,-1,4)$, $B(1,1,2)$, and $C(2,-1,-1)$ are the vertices of a triangle, determine $\overrightarrow{BA}$, $\overrightarrow{BC}$, and $\angle ABC$. Use these to determine the area of $\Delta ABC$. $\left[\frac{\sqrt{14}\times\sqrt{8}\sin(79.1°)}{2}\right]$

13. If $\vec{a}$ and $\vec{b}$ are unit vectors, and $|\vec{a}+\vec{b}|=\sqrt{3}$ determine $(2\vec{a}-5\vec{b})\cdot(\vec{b}+3\vec{a})$. $\left[-\frac{11}{2}\right]$

14. Determine the vector and parametric equation of a line which passes through the point $A(-4,0,3)$ and is parallel to the x-axis.

15. Determine the symmetric equation of a line which passes through the points $A(2,3,-1)$ and $B(5,-2,9)$. $\left[\frac{x-2}{3}=\frac{y-3}{-5}=\frac{z+1}{10}\right]$

16. Determine the Cartesian/Scalar equation of a line which goes through the point $(-3,5)$ that is normal to the line $y=\frac{2}{3}x-7$. $[3x+2y-1=0]$

17. Determine the vector equation of a line which goes through the point $(3,4)$ that is
    a. parallel to the line $y=-\frac{4}{3}x+1$. $[l:[x,y]=(3,4)+t[-3,4]]$
    b. perpendicular to the line $y=2x+5$. $[l:[x,y]=(3,4)+t[2,-1]]$

18. Write the parametric equation of the line that goes through the point $(6,-2,1)$ and is perpendicular to both

$$l_1:[x,y,z]=[1,4,-2]+t[3,-1,1]$$
$$l_2:[x,y,z]=[9,5,-3]+k[1,-3,7]$$

$[x=6-t, y=-2-5t, z=1-2t]$

19. Which of these vector equations represent the same line? $[l_1 \; and \; l_3]$

$$l_1:[x,y,z]=[11,-2,17]+t[3,-1,4]$$
$$l_2:[x,y,z]=[-13,6,-10]+k[-3,1,-4]$$
$$l_3:[x,y,z]=[-7,4,-7]+s[-6,2,-8]$$

20. Given the lines $l_1:[x,y,z]=(3,-7,5)+k[1,-2,4]$ and $l_2$: $[x,y,z]=(-7,-8,0)+m[3,1,-1]$. Determine if the lines intersect. If the lines intersect state the intersection point and determine the acute angle between both lines. $[skew \; lines]$

21. Given lines $l_1:\frac{x-3}{1}=\frac{y+7}{-2}=\frac{5-z}{-4}$ and $l_2:\frac{x+7}{3}=\frac{y+8}{1}=\frac{-z+4}{1}$. Determine if the lines intersect. If the lines intersect state the intersection point and determine the acute angle between both lines.
$[(2,-5,1),\ 78.62°]$

22. Does the line $l_1: [x, y, z] = [-4, 2, -2] + t[2, -1, 3]$
    a. Intersect the z-axis? If so, where? $[0, 0, 4]$
    b. Intersect the y-axis? If so, where? $[no\ intersection]$

23. The position of two helicopters X and Y at time t seconds are given by the formula

$$H_1: [x, y, z] = (11, 3, -3) + t[1, -1, 4]$$
$$H_2: [x, y, z] = (1, -7, -2) + s[2, 1, 9]$$

    a. What is the speed of the two helicopters if distances are measured in metres? $\left[3\sqrt{2}\ and\ \sqrt{86}\right]$
    b. Show that the two helicopters will not collide. $\left[s = \frac{20}{3}, t = \frac{10}{3}\ x\ and\ y\ coordinaes\right]$
    c. Determine the distance between the helicopters when $t = 10$. $\left[\sqrt{2701}\ m\right]$

24. An Enemy Battleship is located at point $B(65, -33)$ notices a stranded Aircraft Carrier located at point $A(-5, 7)$. The Battleship fires a missile towards the Carrier with a velocity of $\vec{v} = [-3.5, 2]\ units/min$. If a Friendly Destroyer located at $D(30, 17)$ notices the missile on sonar 5 minutes after the missile was launched and is able to fire a counter missile with a velocity of $\vec{v}$ = [-1.75, -6.5] $units/min$, how much time do they have before they must fire their counter missile? $[3\ min]$

25. In Question 24, if the Destroyer wanted to have the most accuracy with a missile, they would want to hit the enemy missile at the time when it is closest to the Destroyer.
    a. Determine the time after the enemy missile is shot when the enemy missile is closest to the Destroyer. $[13.6923\ min]$
    b. What is the coordinate of the point of impact? $[(17.07695, -5.6154)]$