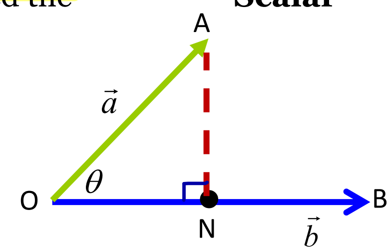
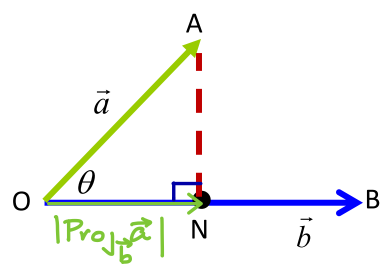
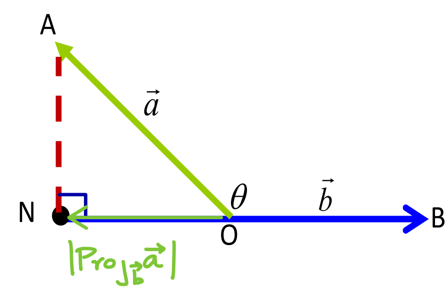
Applications of the Dot Product

Part I. Scalar and Vector Projections

Given two vectors, $\vec{a}$ and $\vec{b}$, placed tail to tail with angle θ between them, drop a perpendicular from the tip of $\vec{a}$ to the line containing $\vec{b}$. The vector lying along the line containing $\vec{b}$, which has magnitude equal to the component of $\vec{a}$ in the direction of $\vec{b}$ (i.e., $\overrightarrow{ON}$ in our diagram), is called the **vector projection** of $\vec{a}$ onto $\vec{b}$. The **magnitude of the vector projection** of $\vec{a}$ onto $\vec{b}$ is called the **Scalar Projection.**



A. Scalar Projections – no direction

$0^\circ \le \theta \le 90^\circ$ **(same direction as $\vec{b}$)**	$90^\circ < \theta \le 180^\circ$ **(opposite direction as $\vec{b}$)**
	

i. Scalar Projection of $\vec{a}$ onto $\vec{b}$

$$\boxed{\left|Proj_{\vec{b}}\vec{a}\right| = \frac{\vec{a}\bullet\vec{b}}{|\vec{b}|}}$$

Note $\quad \left|Proj_{\vec{b}}\vec{a}\right| = \dfrac{|\vec{a}||\vec{b}|\cos\theta}{|\vec{b}|}$

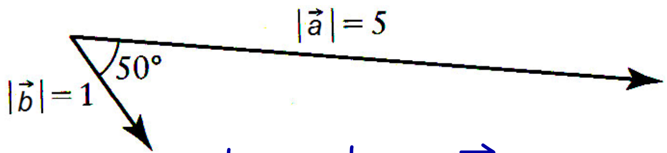
$= \vec{a}\cos\theta$

ii. Scalar Projection of $\vec{b}$ onto $\vec{a}$

$$\left|Proj_{\vec{a}}\vec{b}\right| = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

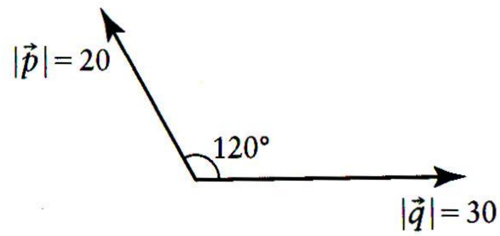
Example 1:

a) Determine the scalar projection of $\vec{a}$ onto $\vec{b}$.



$$|Proj_{\vec b}\vec a| = \frac{\vec a\cdot\vec b}{|\vec b|}$$
$$= \frac{|\vec a||\vec b|\cos\theta}{|\vec b|}$$
$$= (5)\cos 50°$$
$$= 3.21$$

b) Determine the scalar projection of $\vec{p}$ onto $\vec{q}$.



$$|Proj_{\vec q}\vec p| = \frac{\vec p\cdot\vec q}{|\vec q|}$$
$$= |\vec p|\cos\theta$$
$$= (20)\cos 120°$$
$$= -10$$

B. Vector Projections – have direction

The *vector* projection of $\vec{a}$ onto $\vec{b}$ is $\boxed{Proj_{\vec b}\vec{a} = \left(\frac{\vec a\bullet\vec b}{|\vec b|^2}\right)\vec b}$

The *vector* projection of $\vec{b}$ onto $\vec{a}$ is $Proj_{\vec a}\vec b = \left(\frac{\vec a\cdot\vec b}{|\vec a|^2}\right)\vec a$

Example 2: Given vectors $\vec{a} = (-4, 1)$ and $\vec{b} = (4, 3)$, determine $Proj_{\vec a}\vec b$.

$$Proj_{\vec a}\vec b = \left(\frac{\vec a\cdot\vec b}{|\vec a|^2}\right)\vec a$$
$$= \left(\frac{[-4,1]\cdot[4,3]}{\left(\sqrt{(-4)^2+1^2}\right)^2}\right)[-4,1]$$
$$= \left(\frac{-16+3}{17}\right)[-4,1]$$
$$= \frac{-13}{17}[-4,1]$$
$$= \left[\frac{52}{17}, \frac{-13}{17}\right]$$

Example 3: Let $\vec u = proj_{\vec b}\vec a$ where $\vec a = (1,1)$ and $\vec b = (4,2)$. Show that $|\vec a - \vec u| \le |\vec a - k\vec b|$ for all $k \in \mathbb{R}$.

$$\vec u = proj_{\vec b}\vec a$$
$$= \left(\frac{\vec a\cdot\vec b}{|\vec b|^2}\right)\vec b$$
$$= \left(\frac{[1,1]\cdot[4,2]}{\left(\sqrt{4^2+2^2}\right)^2}\right)[4,2]$$
$$= \left(\frac{4+2}{(\sqrt{20})^2}\right)[4,2]$$
$$= \left(\frac{6}{20}\right)[4,2]$$
$$= \left[\frac{6}{5}, \frac{3}{5}\right]$$

$$LS = |\vec a - \vec u|$$
$$= \left|[1,1] - \left[\frac{6}{5}, \frac{3}{5}\right]\right|$$
$$= \left|-\frac{1}{5}, \frac{2}{5}\right|$$
$$= \sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2}$$
$$= \frac{\sqrt5}{5}$$

$$RS = |\vec a - k\vec b|$$
$$= |[1,1] - k[4,2]|$$
$$= |[1-4k, 1-2k]|$$
$$= \sqrt{(1-4k)^2 + (1-2k)^2}$$
$$= \sqrt{1-8k+16k^2+1-4k+4k^2}$$
$$= \sqrt{20k^2-12k+2}$$
$$= \sqrt{2(10k^2-6k+1)}$$
$$= \sqrt2\left(\sqrt{10k^2-6k+1}\right), \quad \underbrace{\sqrt{10k^2-6k+1}}_{\text{minimum value is } \sqrt{\frac{1}{10}}} > 0$$

$$\therefore RS \geq \sqrt2\left(\sqrt{\frac{1}{10}}\right)$$
$$\geq \frac{\sqrt5}{5}$$

$$|\vec a - \vec u| \leq |\vec a - k\vec b|$$

$y = 10k^2 - 6k + 1$

$y = 10\left(k^2 - \frac{3}{5}k + \frac{9}{100} - \frac{9}{100}\right) + 1$

$y = 10\left(k - \frac{3}{10}\right)^2 + \frac{1}{10}$

min value is $\frac{1}{10}$

Example 4: The scalar projection of vector $\vec{u} = [1, m, 0]$ onto vector $\vec{v} = [2,2,1]$ is 4. Determine the value of m.

$$\left|Proj_{\vec{v}}\vec{u}\right| = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

$$4 = \frac{[1,m,0]\cdot[2,2,1]}{\sqrt{2^2+2^2+1}}$$

$$4 = \frac{2+2m}{3}$$

$$12 = 2+2m$$

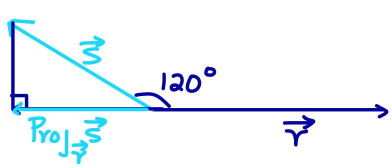
$$2m = 10$$

$$\boxed{m=5}$$

Example 5: The vector $\vec{r}$ is twice as long as the vector $\vec{s}$. The angle between the vectors is 120°. The vector projection of $\vec{s}$ on $\vec{r}$ is [2, –1, 7]. Determine $\vec{r}$.

Given $|\vec{r}| = 2|\vec{s}|$

$|\vec{s}| = \frac{1}{2}|\vec{r}|$



$$Proj_{\vec{r}}\vec{s} = \left(\frac{\vec{r}\cdot\vec{s}}{|\vec{r}|^2}\right)\vec{r}$$

$$[2,-1,7] = \frac{|\vec{r}||\vec{s}|\cos 120^\circ}{|\vec{r}|^2}\vec{r}$$

$$= \frac{|\vec{r}|\frac{|\vec{r}|}{2}\left(-\frac{1}{2}\right)}{|\vec{r}|^2}\vec{r}$$

$$= -\frac{1}{4}\vec{r}$$

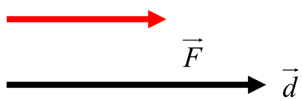
$$\vec{r} = [-8,4,-28]$$

Part II. Work

Definition: In Physics, WORK is done whenever a force, applied to an object, causes a displacement in the object from one position to another.

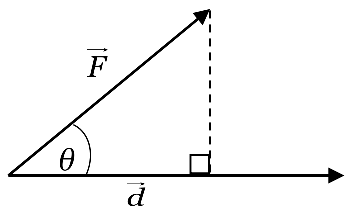
- WORK is equal to the displacement traveled multiplied by the magnitude of the applied force in the direction of motion.

For instance, if the force is in the same direction as the displacement, then just multiply the magnitudes.



$$work = |\vec{F}||\vec{d}|$$

However if the force acts at an angle to the [redacted] displacement vector, we use the component of the force, in the direction of the displacement vector (i.e we use the projection of $\vec{F}$ onto $\vec{d}$)



$$\boxed{\begin{aligned} work &= |\vec{F}||\vec{d}|\cos\theta \\ &= \vec{F}\cdot\vec{d} \end{aligned}}$$

The work done on an object is the dot product of the force applied on the object, and the displacement of the object.

Note: - Work is a scalar quantity. The unit of measurement is the **Joule (J)** or Newton-metre

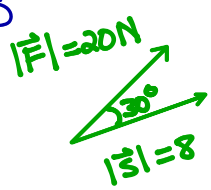
(N-m).

- No matter how much force is applied, if no displacement occurs, work = 0.

Example 1: A crate, on a ramp is hauled 8m up the ramp by a constant force of 20N applied at an angle of 30° to the ramp. Calculate the work done by the force.

Example2: A shopper in a supermarket pushes a cart with a force of 35 N directed at an angle of 25° downward from the horizontal. Find the work done by the shopper as she moves down an aisle 50 m in length.

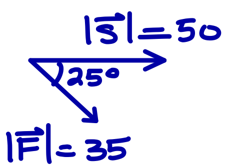
Example ①



$$\begin{aligned} W &= \vec{F}\cdot\vec{s} \\ &= |\vec{F}||\vec{s}|\cos\theta \\ &= (20)(8)\cos 30° \\ &= 139 \end{aligned}$$

The work done is approx. 139 J.

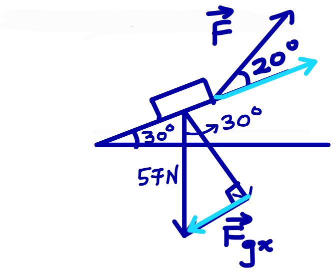
Example ②



$$\begin{aligned} W &= \vec{F}\cdot\vec{s} \\ &= |\vec{F}||\vec{s}|\cos\theta \\ &= (35)(50)\cos 25° \\ &= 1586 \end{aligned}$$

The work done is approx. 1586 J

Example 3: A crate with a weight of 57 N rests on a frictionless ramp inclined at an angle of 30° to the horizontal. What force must be applied at an angle of 20° to the ramp so that the crate remains at rest?



Let $\vec{F}_{gx}$ be horizontal vector component of the 57 N force, $\vec{F}$ be the force applied and $\vec{F}_H$ be the horizontal component of $\vec{F}$.

$$|\vec{F}_{gx}| = 57\sin 30^\circ$$
$$= 28.5$$

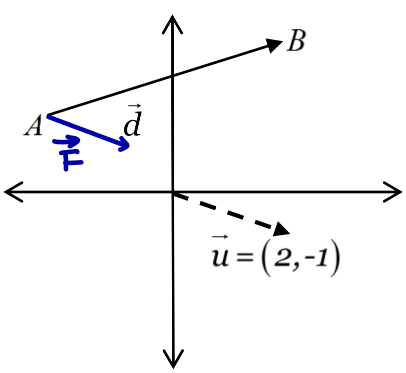
At equilibrium,
$$|\vec{F}_H| = |\vec{F}_{gx}|$$

$$\cos 20^\circ = \frac{|\vec{F}_H|}{|\vec{F}|}$$

$$|\vec{F}| = \frac{28.5}{\cos 20^\circ}$$
$$= 30.3$$

A force of 30.3 N must be applied so that the crate remains at rest

Example 4: A force 15N acting along the vector $\vec{u} = (2,-1)$, displaces a particle from A(-4, 2) to B(1, 5). If the distance is in meters, calculate the amount of work done.



$$\vec{d} = \vec{AB}$$
$$= \vec{OB} - \vec{OA}$$
$$= [1,5] - [-4,2]$$
$$= [5,3]$$

$$\vec{F} = 15\hat{u}$$
$$= \frac{15\vec{u}}{|\vec{u}|}$$
$$= \frac{15[2,-1]}{\sqrt{(2)^2+(-1)^2}}$$
$$= \left[\frac{30}{\sqrt{5}}, \frac{-15}{\sqrt{5}}\right]$$
$$= [6\sqrt{5}, -3\sqrt{5}]$$

$$W = \vec{F} \cdot \vec{d}$$
$$= [6\sqrt{5}, -3\sqrt{5}] \cdot [5,3]$$
$$= 30\sqrt{5} - 9\sqrt{5}$$
$$= 21\sqrt{5}$$
$$= 46.957$$

Work done is ~ 47.0 J

Practice

1. An object is dragged 5m up a ramp under a constant force of 30N applied at an angle 30° to the ramp. Find the work done.

2. A man in a wheelchair moves 15m down a ramp inclined at an angle of 10° to the horizontal. The mass of the man and the wheelchair together is 80kg. (1kg = 9.8N) Calculate the work done.

3. An object is dragged 5m on level ground by a 20N force that is applied 50° to the ground. It is then dragged 8m up a ramp with the same force. The inclination of the ramp is 30° to the ground. At the top of the ramp, the object is dragged, with the same force, horizontally 13m. Find the total work done.

4. A box is lifted through a distance of 1.2 m and placed on a wagon by exerting a force of 105 N. The wagon is then pulled through a distance of 25 m by a 45 N force applied at an angle of 35° to the ground. Find the total work done.

5. Determine the work done by a force of magnitude 55N acting in the direction of the vector $\vec{u} = (2,-2,1)$, which moves an object from A(1,4,-1) to B (-1 ,2,1).The distance is in metres.

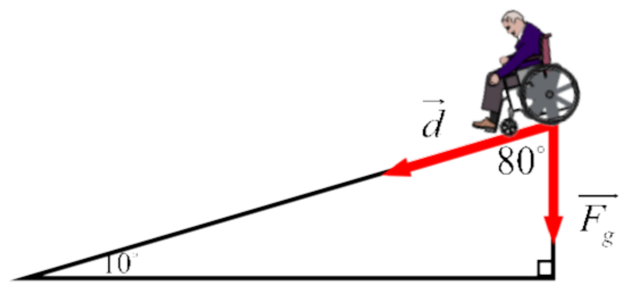
Practice

1. An object is dragged 5m up a ramp under a constant force of 30N applied at an angle 30° to the ramp. Find the work done.

$$\mathbf{W} = |\vec{\mathbf{F}}||\vec{\mathbf{d}}|\cos(\theta)$$
$$= (30)(5)\cos(30^\circ)$$
$$= 129.9\,\mathbf{J}$$

2. A man in a wheelchair moves 15m down a ramp inclined at an angle of 10° to the horizontal. The mass of the man and the wheelchair together is 80kg. (1kg = 9.8N) Calculate the work done.

$$\mathbf{W} = |\vec{\mathbf{F}}_g||\vec{\mathbf{d}}|\cos 80^\circ$$
$$\mathbf{W} = 80 \times 9.8 \times 15\cos 80^\circ$$
$$\mathbf{W} = 2042\ \mathbf{J}$$



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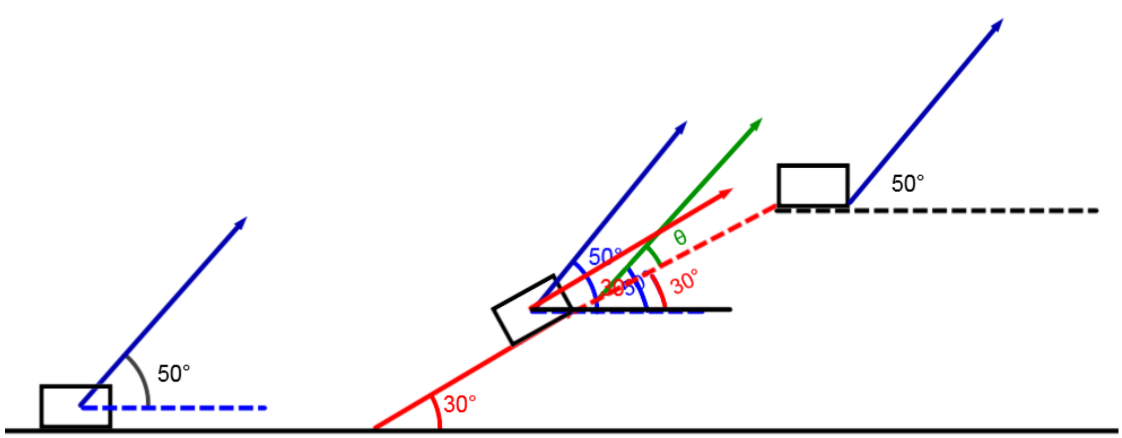
work done before and after ramp : $18 \times 20 \times \cos 50^\circ = 231.40$ J

work done on ramp : $8 \times 20 \times \cos 20^\circ = 150.35$ J

Total work done : 381.75 J



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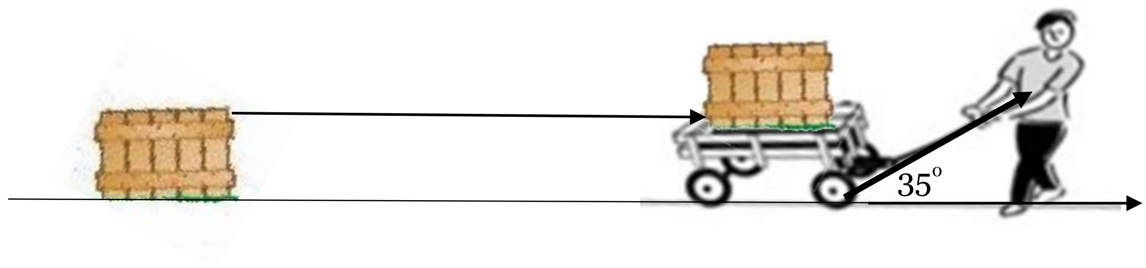


$$W_1 = |\vec{F}||\vec{d}| = (105)(1.2) = 126\,J$$

$$W_2 = |\vec{F}||\vec{d}|\cos(35^\circ) = (45)(25)(0.8191) = 921.55\,J$$

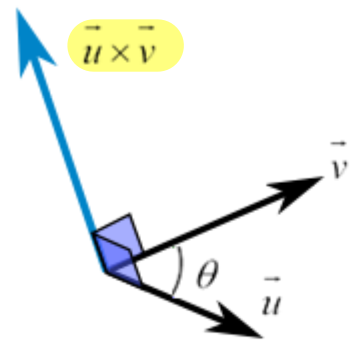
$$W_{Total} = W_1 + W_2 = 126 + 921.55 = 1047.55J$$

5. Determine the work done by a force of magnitude 55N acting in the direction of the vector $\vec{u} = [2,-2,1]$, which moves an object from A(1,4,-1) to B (-1 ,2,1).The distance is in metres.

$$\vec{F} = k\vec{u}$$
$$\vec{F} = k[2,-2,1]$$
$$|\vec{F}| = k\sqrt{(2)^2+(-2)^2+(1)^2}$$
$$55 = 3k$$
$$k = \frac{55}{3} \Rightarrow \vec{F} = \frac{55}{3}[2,-2,1]$$

$$\vec{d} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = [-2,-2,2]$$
$$W = \vec{F} \bullet \vec{d}$$
$$W = \frac{55}{3}[2,-2,1] \bullet [-2,-2,2]$$
$$W = \frac{110}{3}J$$
$$W \doteq 36.7J$$

Cross Product of 2 Vectors $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ – in R^3



- not multiply, slightly bigger
- also known as vector product
- result is always a vector not a scalar
- cross product is a particular vector that's perpendicular to 2 non-collinear vectors, in fact, **there's an infinite number of such vectors!**

Cross Product – Algebraic Vectors

Given the vectors $\vec{\mathbf{u}} = [\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}]$ and $\vec{\mathbf{v}} = [\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}]$ then:

1) Set up the vector components in the following manner:

For $\vec{u} \times \vec{v}$:
$$\begin{matrix} \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} & \mathbf{u_1} \\ \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_1} \end{matrix}$$

2) To determine the x, y, and z component:

x-component of $\vec{u} \times \vec{v}$, conduct the following operation on the **middle** four terms:

Determinant of a 2x2 matrix

$$\begin{vmatrix} \mathbf{u_2} & \mathbf{u_3} \\ \mathbf{v_2} & \mathbf{v_3} \end{vmatrix} = \mathbf{u_2 v_3 - u_3 v_2}$$

"down product - up product"

y-component: conduct the same operation, but on the four terms on the **right**:

$$\begin{vmatrix} \mathbf{u_3} & \mathbf{u_1} \\ \mathbf{v_3} & \mathbf{v_1} \end{vmatrix} = \mathbf{u_3 v_1 - u_1 v_3}$$

z-component: repeat for the four terms on the **left**:

$$\begin{vmatrix} \mathbf{u_1} & \mathbf{u_2} \\ \mathbf{v_1} & \mathbf{v_2} \end{vmatrix} = \mathbf{u_1 v_2 - u_2 v_1}$$

Determinant method:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{u_1} & \mathbf{u_2} & \mathbf{u_3} \\ \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} \end{vmatrix}$$

Method: $\vec{u} \times \vec{v}$

1) Write the vectors twice ($\vec{u}$ above $\vec{v}$)
2) Eliminate the first and last column.
3) "down" products — "up" products as shown.

$$\begin{matrix} \cancel{u_1} & u_2 & u_3 & u_1 & u_2 & \cancel{u_3} \\ \cancel{v_1} & v_2 & v_3 & v_1 & v_2 & \cancel{v_3} \end{matrix}$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = [u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1]$$
$$= \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

Derivation is on p. 405 MHR

Example 1: Find the cross product of $\vec{u}$ = [-2, 1, -4] and $\vec{v}$ =[3, 0, -1].

$$\vec{u} \times \vec{v} = [-2,1,-4] \times [3,0,-1]$$
$$= [-1-0, -12-2, 0-3]$$
$$= [-1,-14,-3]$$

$$\begin{array}{cccccccc} -2 & 1 & -4 & -2 & 1 & -4 \\ 3 & 0 & -1 & 3 & 0 & -1 \end{array}$$

Example 2: Find the cross product of $\vec{u}$ = [-2, 1, -4] and $\vec{v}$ =[6, -3, 12].

$$\vec{u} \times \vec{v} = [-2,1,-4] \times [6,-3,12]$$
$$= [12-12, -24+24, 6-6]$$
$$= [0,0,0]$$

$$\begin{array}{cccccc} -2 & 1 & -4 & -2 & 1 & -4 \\ 6 & -3 & 12 & 6 & -3 & 12 \end{array}$$

Note $[6,-3,12]$
$= -3[-2,1,-4]$
$\vec{v} = -3\vec{u}$ (collinear vectors)

$$\boxed{\vec{u} \times \vec{v} = \vec{0} \iff \vec{u} = k\vec{v}}$$

Example 3: If $\vec{a} = [1,3,-1]$, $\vec{b} = [2,1,5]$ and $\vec{v} = [-3,y,z]$ $\vec{a} \times \vec{v} = \vec{b}$, find $\boldsymbol{y}$ and $\boldsymbol{z}$.

$\vec{a} \times \vec{v}$
$= [1,3,-1] \times [-3,y,z]$
$= [3z+y, 3-z, y+9]$

$$\begin{array}{cccccc} 1 & 3 & -1 & 1 & 3 & -1 \\ -3 & y & z & -3 & y & z \end{array}$$

$\vec{a} \times \vec{v} = \vec{b}$
$[3z+y, 3-z, y+9] = [2,1,5]$
$3z+y = 2$ —— 1

$3-z = 1$
$z = 3-1$
$\boxed{z=2}$

$y+9 = 5$
$y = 5-9$
$\boxed{y=-4}$

Check
Sub in 1
LS $= 3(2)+(-4)$ RS $=2$
$=2$
$\therefore$ LS = RS

z is 2 and y is -4

Properties of the Cross Product

The Cross Product is:

1) **Anti-Commutative**: $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$

2) **Distributive** over vector addition: $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$

3) **Associative** over scalar multiplication: $m(\vec{u}\times\vec{v}) = (m\vec{u})\times\vec{v} = \vec{u}\times(m\vec{v})$, $m \in \mathbb{R}$

4) If $\vec{u}$ and $\vec{v}$ are non-zero, $\vec{u}\times\vec{v} = \vec{0}$ if and only if $\vec{u}$ and $\vec{v}$ are collinear.

Example 4: If ABC is a triangle with vertices A(1, 1, -1), B(1, 0, 1), and C(1 + *k*, 0, 2) and $\overrightarrow{AB}\times\overrightarrow{AC} = [-1,2,1]$, find the value of *k*.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= [1,0,1]-[1,1,-1]$
$= [0,-1,2]$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= [1+k,0,2]-[1,1,-1]$
$= [k,-1,3]$

$\begin{matrix} 0 & -1 & 2 & 0 & -1 & 2 \\ k & -1 & 3 & k & -1 & 3 \end{matrix}$

$\overrightarrow{AB}\times\overrightarrow{AC} = [-1,2,1]$
$[-3+2, 2k-0, 0+k] = [-1,2,1]$
$[-1,2k,k] = [-1,2,1]$

$2k = 2$
$\boxed{k=1}$

The value of k is 1

Example 5: Determine the value of m and n for $\vec{a} = [m,-12,9]$ and $\vec{b} = [5,n,-3]$ such that

$\vec{a}\times\vec{b} = \vec{0}$. What is the relationship between $\vec{a}$ and $\vec{b}$.

$\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a} = k\vec{b}$, $k \in \mathbb{R}$
$[m,-12,9] = k[5,n,-3]$
$m = 5k$ — ①
$-12 = kn$ — ②
$9 = -3k$
$\boxed{k=-3}$

Sub in ①
$m = 5(-3)$
$\boxed{m=-15}$

Sub in ②
$-12 = (-3)n$
$\boxed{n=4}$

$\vec{a}$ and $\vec{b}$ are scalar multiples of each other
They are Collinear vectors

$\begin{matrix} m & -12 & 9 & m & -12 & 9 \\ 5 & n & -3 & 5 & n & -3 \end{matrix}$

Method ②
$\vec{a}\times\vec{b} = \vec{0}$
$[m,-12,9]\times[5,n,-3] = [0,0,0]$
$[36-9n, 45+3m, mn+60] = [0,0,0]$

$36-9n=0$
$n = \frac{36}{9}$
$\boxed{n=4}$

$45+3m=0$
$3m=-45$
$m=-\frac{45}{3}$
$\boxed{m=-15}$

check ③:
LS $= mn+60$
$=(-15)(4)+60$
$=0$

RS $=0$

LS = RS

Magnitude of the Cross Product

The magnitude of the cross product is defined according to the following equation:

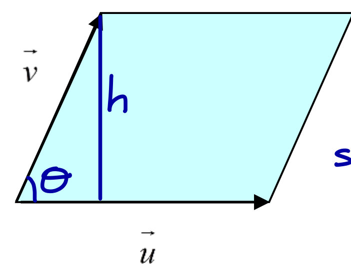
$$\boxed{|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta}$$

, where θ is the angle between the vectors such that $0° \le \theta \le 180°$.

It represents the area of the parallelogram enclosed by the two vectors.

Area of parallelogram = base × height
$= |\vec{u}||\vec{v}|\sin\theta$

$$\mathbf{A}_{\mathbf{parallelogram}} = |\vec{\mathbf{u}}\times\vec{\mathbf{v}}|$$



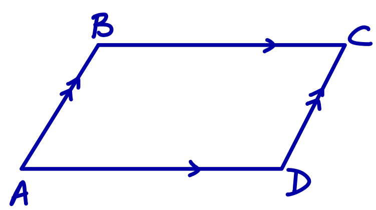
base $= |\vec{u}|$

Note
$\sin\theta = \dfrac{h}{|\vec{v}|}$
$\therefore h = |\vec{v}|\sin\theta$

Example 6: Three vertices of a **parallelogram ABCD** are **A(3,-1,2), B(1,2,-4)** and **C(-1,1,2)**.

a) find the coordinate of the fourth vertex.

a) find the area of triangle ABC.



a) $\vec{AB} = \vec{DC}$
$\vec{OB} - \vec{OA} = \vec{OC} - \vec{OD}$
$[1,2,-4] - [3,-1,2] = [-1,1,2] - \vec{OD}$
$[-2,3,-6] = [-1,1,2] - \vec{OD}$
$\vec{OD} = [-1,1,2] - [-2,3,-6]$
$\vec{OD} = [1,-2,8]$

Coordinates of D are $(1,-2,8)$

b) Area $\Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AD}|$

$\vec{AB} = [-2,3,-6]$
$\vec{AD} = \vec{OD} - \vec{OA}$
$= [1,-2,8] - [3,-1,2]$
$= [-2,-1,6]$

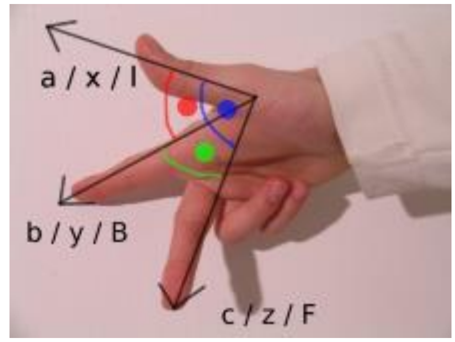
$\vec{AB}\times\vec{AD}$
$= [12,24,8]$
$= 4[3,6,2]$

$$\begin{array}{cccccccc} -2 & 3 & -6 & -2 & 3 & -6 \\ -2 & -1 & 6 & -2 & -1 & 6 \end{array}$$

Area $\Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AD}|$
$= \frac{1}{2}\left(4\sqrt{3^2+6^2+2^2}\right)$
$= 2\sqrt{49}$
$= 14$

Area of ΔABC is 14 units^2

Direction of the Cross Product – Into the Page or Out of the Page



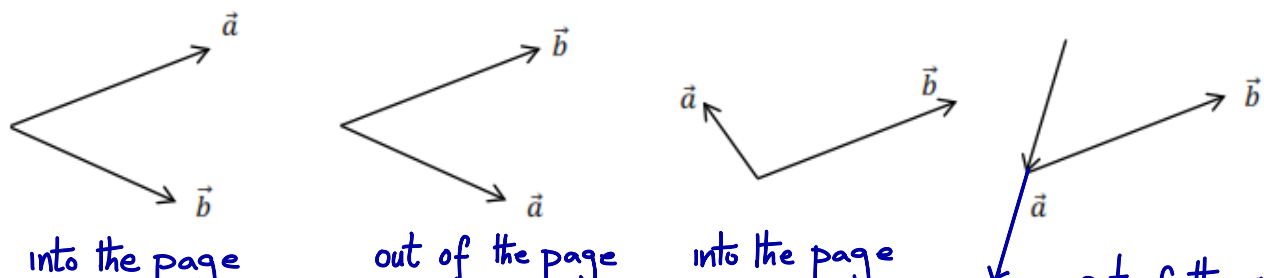
Recall that the cross product gives us a vector that is perpendicular to two vectors. To determine whether the cross product $\vec{a} \times \vec{b}$ is into the page or out of the page, we use the Right Hand Rule.

- Make your thumb lie on the first vector ($\vec{a}$).
- Make your index finger lie on the second vector($\vec{b}$)
- Make your middle finger perpendicular to your thumb

Example 7: Given the following vectors determine if the cross product is into the page or out of the page.

- NOTE: Just like the dot product, **vectors must be tail to tail** when evaluating a cross product.

$\vec{a} \times \vec{b}$



into the page | out of the page | into the page | out of the page

Example 8: Determine the following cross products using the correct sign convention

a) $\hat{i} \times \hat{j} =$ $\hat{k}$

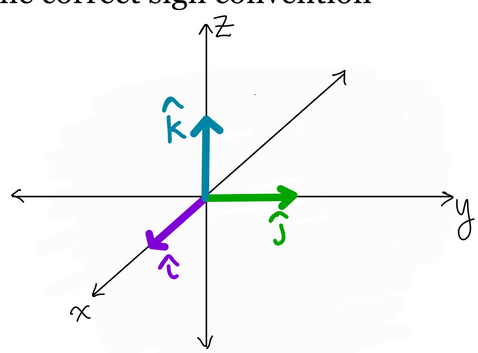
b) $\hat{k} \times \hat{i} =$ $\hat{j}$

c) $\hat{i} \times \hat{k} =$ $-\hat{j}$

d) $\hat{k} \times \hat{j} =$ $-\hat{i}$

e) $\hat{j} \times \hat{i} =$ $-\hat{k}$

f) $\hat{j} \times \hat{k} =$ $\hat{i}$



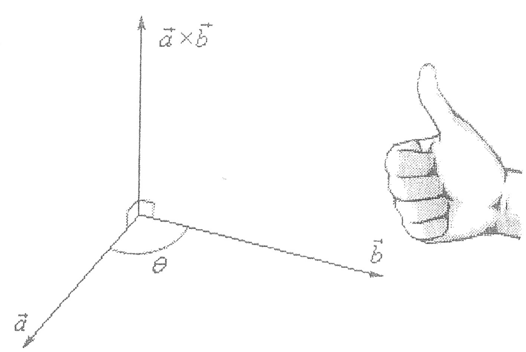
$\hat{i} = [1,0,0]$ $\hat{j} = [0,1,0]$

$\hat{i} \times \hat{j}$

$= [1,0,0] \times [0,1,0]$

$= [0,0,1]$

$= \hat{k}$



Curl the fingers of your right hand from the first vector to the second. The thumb then points in the direction of the cross product of the two vectors.

Theorem:

If θ is the angle between $\vec{a}$ and $\vec{b}$ (so $0 \le \theta \le \pi$), then $\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \, sin\theta$

Proof:

If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, then the cross product of $\vec{a}$ and $\vec{b}$ is the vector

$\vec{a}\times\vec{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$

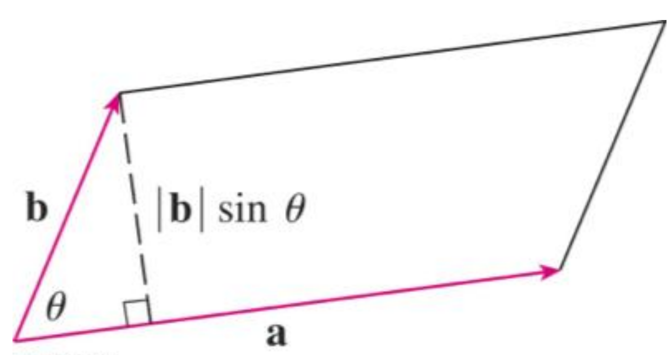
Therefore:

$$
\begin{aligned}
\left|\vec{a}\times\vec{b}\right|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - (\vec{a}\bullet\vec{b})^2 \\
&= \left|\vec{a}\right|^2\left|\vec{b}\right|^2 - \left|\vec{a}\right|^2\left|\vec{b}\right|^2 cos^2\theta \\
&= \left|\vec{a}\right|^2\left|\vec{b}\right|^2 (1 - cos^2\theta) \\
&= \left|\vec{a}\right|^2\left|\vec{b}\right|^2 sin^2\theta
\end{aligned}
$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$, because $\sin\theta \ge 0$ when $0 \le \theta \le \pi$, we have:

$$\left|\vec{a}\times\vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \, sin\theta$$

The geometric interpretation of this theorem can be seen from this figure.



If $\vec{a}$ and $\vec{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\vec{a}|$ and altitude $|\vec{b}|$ sin θ, and area

$$A = (|\vec{b}| \sin\theta)|\vec{a}| = |\vec{a} \times \vec{b}|$$

Thus, we have the following way of interpreting the magnitude of a cross product. The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

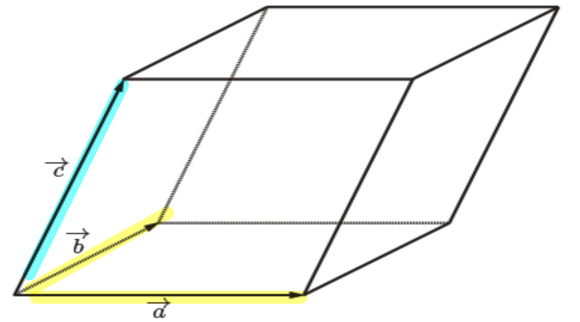
Applications of the Dot Product and Cross Product

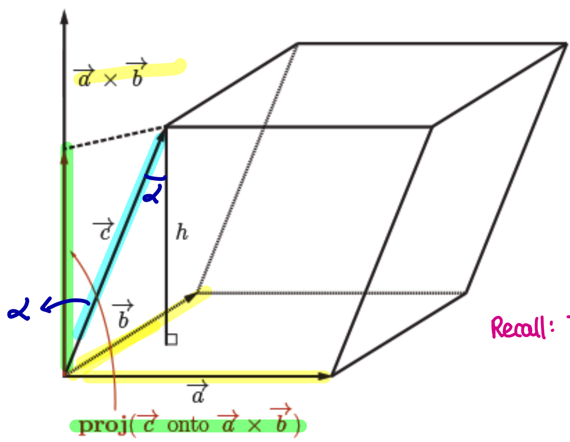
I. Volume of Parallelepiped

A **parallelepiped** is a box-like solid, where the opposite faces of which are parallel and congruent parallelograms.

Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors whose tails meet at one vertex of the parallelepiped.

The absolute value of the triple scalar product of these three vectors gives the volume of the parallelepiped.





$\cos\alpha = \frac{h}{|\vec{c}|}$

$h = |\vec{c}|\cos\alpha$

Note: α is the angle between $\vec{c}$ and the cross product $\vec{a}\times\vec{b}$, which is also perpendicular to the base

$V = (\text{area of base})(\text{height})$

$= |\vec{a}\times\vec{b}||\vec{c}|\cos\alpha$

Recall: $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$= |\vec{c}\cdot\vec{a}\times\vec{b}|$ ← Triple Scalar Product

Volume = (area of base) × height

= (area of parallelogram) ×height

parallelogram is made up of vector $\vec{a}$ and $\vec{b}$ so its area = $|\vec{a}\times\vec{b}|$

The height = the magnitude of the projection of $\vec{c}$ onto the vector perpendicular to the base: $\boxed{\left|Proj_{\vec{a}\times\vec{b}}\vec{c}\right| = \frac{\vec{c}\bullet\left(\vec{a}\times\vec{b}\right)}{\left|\vec{a}\times\vec{b}\right|}}$

Volume = (area of parallelogram) x height

$$\boxed{V = \left|\vec{c}\bullet\left(\vec{a}\times\vec{b}\right)\right|}$$

Question: *Is* $|\vec{c}\bullet(\vec{a}\times\vec{b})|$ *equivalent to* $|\vec{a}\bullet(\vec{b}\times\vec{c})|$? YES.

Triple Scalar Product: is called the quantity $\vec{c}\bullet(\vec{a}\times\vec{b})$, since it returns a scalar value.

Definition: Three vectors $\vec{a}, \vec{b}$, *and* $\vec{c}$ are coplanar if and only if $\vec{c}\bullet(\vec{a}\times\vec{b})=0$

Example 9: Determine the volume of a parallelepiped given the vectors $\vec{a}=[-2,2,5]$, $\vec{b}=[0,4,1]$ and $\vec{c}=[0,5,-1]$.

$$\begin{matrix} 0 & 4 & 1 & 0 & 4 & 1 \\ 0 & 5 & -1 & 0 & 5 & -1 \end{matrix}$$

$$V = |\vec{a}\cdot\vec{b}\times\vec{c}|$$
$$= |[-2,2,5]\cdot[0,4,1]\times[0,5,-1]|$$
$$= |[-2,2,5]\cdot[-9,0,0]|$$
$$= 18$$

Volume is 18 units^3

Example 10: Determine if the vectors [1,3,2],[5,0,−1], and [−4,3,3] are coplanar.

$$\begin{matrix} 5 & 0 & -1 & 5 & 0 & -1 \\ -4 & 3 & 3 & -4 & 3 & 3 \end{matrix}$$

$$[1,3,2]\cdot[5,0,-1]\times[-4,3,3]$$
$$= [1,3,2]\cdot[0+3,4-15,15-0]$$
$$= [1,3,2]\cdot[3,-11,15]$$
$$= 3-33+30$$
$$= 0$$

Since the triple scalar product is zero, the three vectors are coplanar

Example 11. Circle whether the following expressions are vectors, scalar, or meaningless.

a) $(\vec{a}+\vec{b})\bullet(\vec{a}+\vec{c})$ vector scalar meaningless

b) $\dfrac{(\vec{a}\bullet\vec{b})\vec{c}\times(\vec{a}\times\vec{b})}{|\vec{c}|}$ vector scalar meaningless

c) $\vec{a}\times\vec{b}+\vec{u}\bullet\vec{c}$ vector scalar meaningless

d) $\dfrac{\vec{a}\bullet\vec{b}+\vec{a}\bullet(\vec{a}\times\vec{b})}{|\vec{b}|}$ vector scalar meaningless

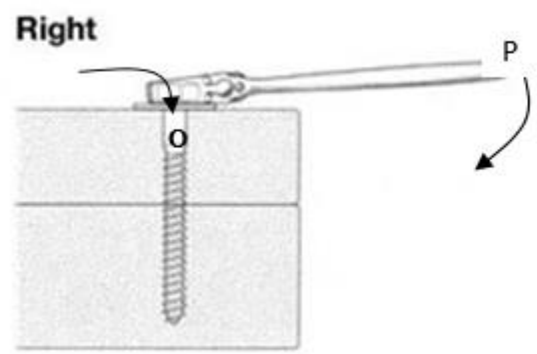
II. Torque

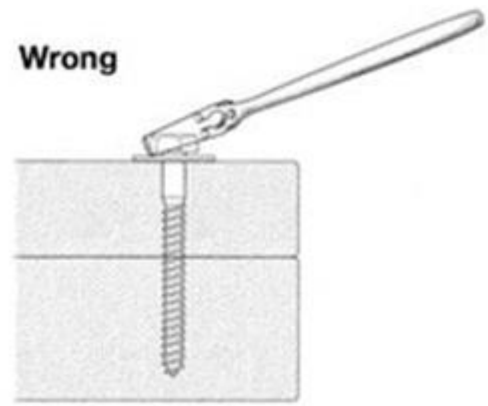
Torque

Definition: Torque can be considered the turning effect of a force on an object.

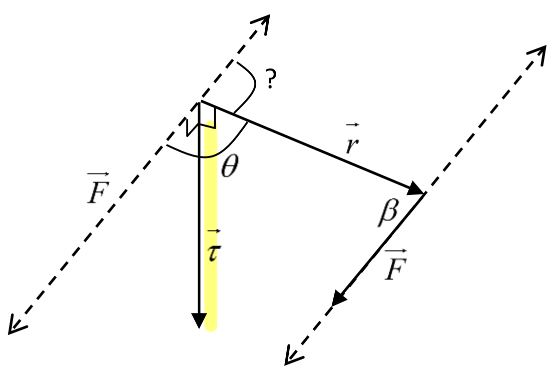
It has a magnitude, measured in Newton-metres (Nm) and a direction. It is therefore, a vector value.

For example: Turning a bolt with a wrench to drive it into a block of wood.





To find θ, we arrange $\vec{r}$ and $\vec{F}$ tail to tail.



Note: Torque is max when θ = 90°
The most efficient way to maximize torque with a certain force is to maximize $\vec{r}$ AND/OR $\sin(\theta)$

The torque vector will always act in a direction **perpendicular** to both $\vec{r}$ and $\vec{F}$.

In our example the bolt is either being pushed into the block of wood (moving into the board) or being pulled out of the block of wood (moving out of the board). In both cases, the motion is orthogonal to the applied force and to the lever arm.

(I) Therefore, the torque produced can be determined by finding the cross product of $\vec{r}$ and $\vec{F}$.

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

Greek letter tau (τ)

(II) It follows then that the magnitude of the torque produced is the magnitude of the cross product of $\vec{r}$ and $\vec{F}$. r and

$$\left|\vec{\tau}\right| = \left|\vec{r} \times \vec{F}\right| \quad \text{or} \quad \left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$

***Recall:** θ is the angle between the $\vec{r}$ and $\vec{F}$ when arranged tail to tail.

Its magnitude measures the twisting effect of the force, while its direction gives the direction of the axis through O about which the force tends to twist (i.e. down and clockwise with a right-hand thread or use the right-hand rule.

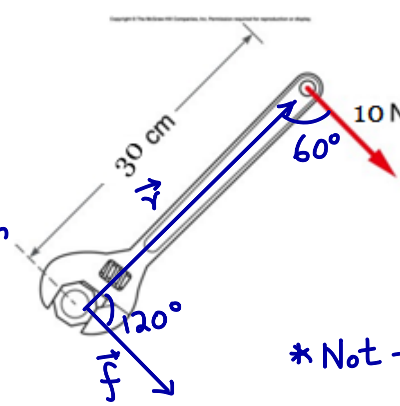
The direction of the torque vector is found by using the right hand rule.

***Terminology: i. Orthogonal ii. Normal iii. Fulcrum iv. "tighten bolt" vs. "loosen bolt"** (into/out of paper)

Example1: A 10 N force is applied at the end of a 30cm wrench with which it makes a 60° angle. Calculate the magnitude of the torque.

$$\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$
$$= (0.3)(10)\sin 120^\circ$$
$$\doteq 2.60$$

The magnitude of the torque is 2.60 Nm

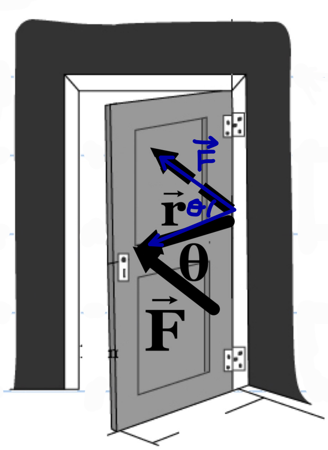


*Not to scale

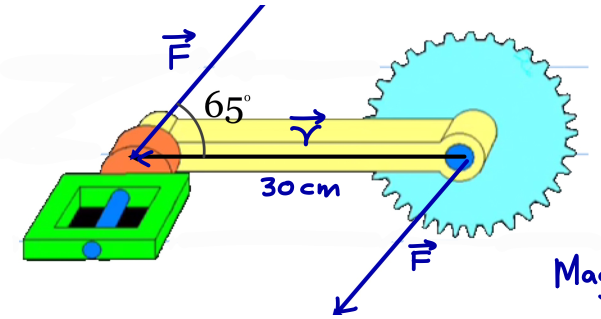
Example 2: A 50N force is applied at a point on a door that is 70cm from the side of the hinged edge. The force makes a 30° angle with the door. Calculate the magnitude and describe the direction of the torque vector. Include a diagram.

$$\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin\theta$$
$$= (0.7)(50)\sin 30^\circ$$
$$= 17.5$$

The magnitude of the torque is 17.5 Nm and the direction is into the page



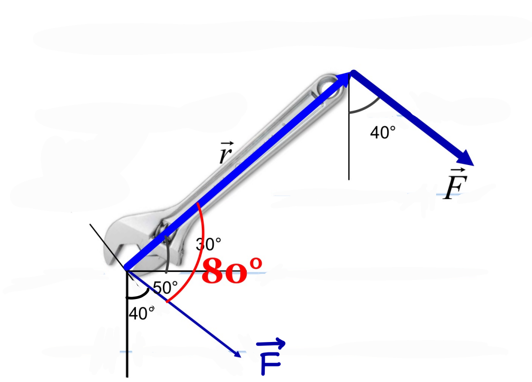
Example3: A force of 50N is applied to a bike pedal making a 65° angle with the lever arm. If the lever arm is 30cm long, calculate the magnitude of the torque produced.



$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$
$$= (0.3)(50)\sin 65^\circ$$
$$= 13.6$$

Magnitude of the torque is 13.6 Nm

Example4: A bolt is being rotated by a 20cm wrench. If the wrench is oriented 30° to the horizontal, and a downward force of 30N is being applied to the end of the wrench at an angle 40° to the vertical, find the magnitude of the torque produced. Is the bolt being screwed in or removed?



$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$
$$= (0.2)(30)\sin 80^\circ$$
$$= 5.91$$

The torque produced is 5.91 J The bolt is being screwed in

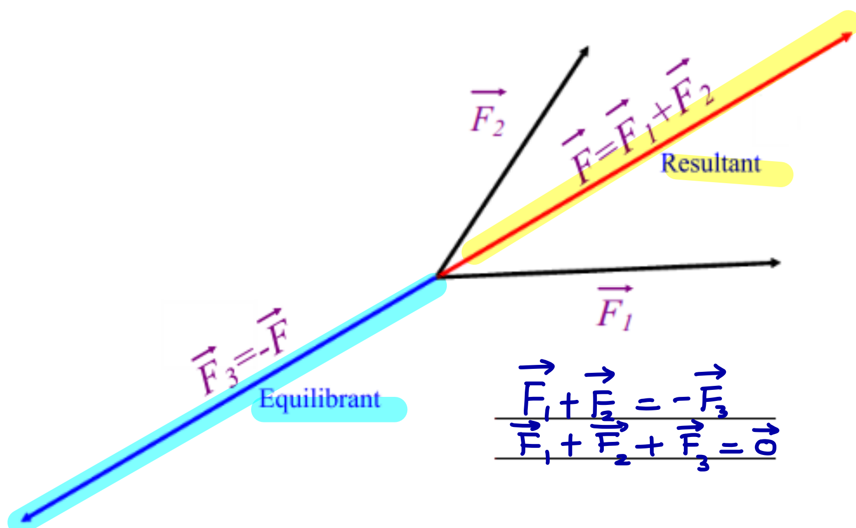
Applications of Vector Addition – Force

Force : A physical influence that causes a change in direction on a physical object. It is measured in a unit called Newtons (N).

To describe a force it is necessary to state:
i) its direction
ii) the point at which it is applied
iii) its magnitude

The *resultant* is the sum of the vectors representing two or more forces.
The ***equilibrant*** is the opposite force that would exactly counterbalance the resultant.

Equilibrant Force: Let $\vec{F_1}$ and $\vec{F_2}$
be two forces acting upon an object. The resultant vector can be represented by a third vector using the concepts from vector addition.

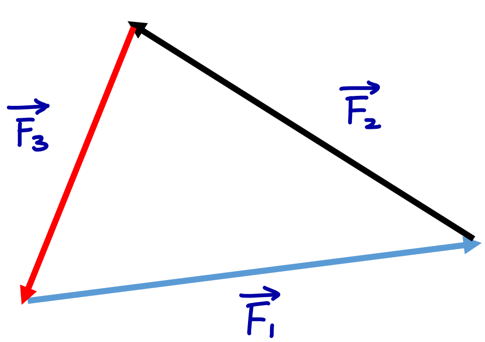


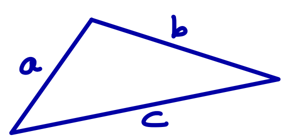
$\vec{F}_1+\vec{F}_2=-\vec{F}_3$

$\vec{F}_1+\vec{F}_2+\vec{F}_3=\vec{0}$

For a system of three forces to being at equilibrium the vectors representing those forces, when placed head (or tip) to tail.

Triangle Inequality theorem Any side of a triangle must be shorter than the other 2 sides added together





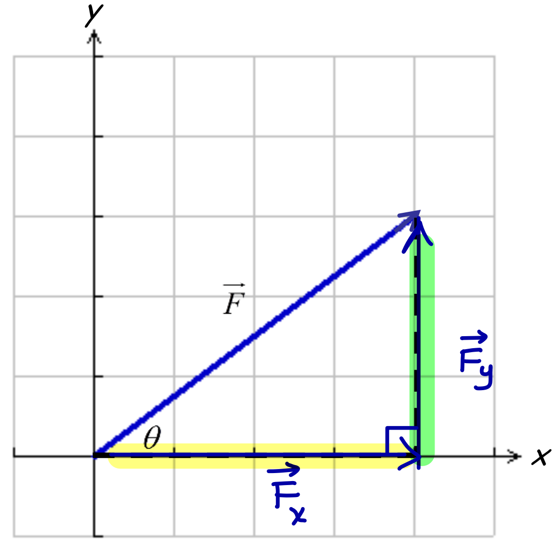
$a < b + c$
$b < a + c$
$c < a + b$

Note $c = a + b$ (straight line)
$c > a + b$ (No triangle can be formed)

System at equilibrium $\Rightarrow |\vec{F}_3| \leq |\vec{F}_1| + |\vec{F}_2|$

RESOLVING VECTORS INTO COMPONENTS (used in application problems when a Cartesian coordinate system is not used)

A force can be resolved into horizontal and vertical components. The magnitudes are as follows



Horizontal Component

$$\boxed{|\vec{F}_x| = |\vec{F}|\cos(\theta)}$$

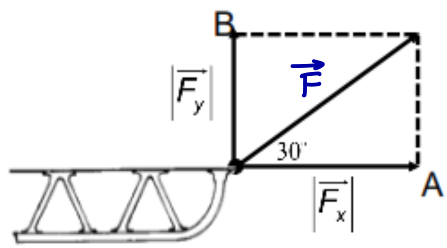
Vertical component

$$\boxed{|\vec{F}_y| = |\vec{F}|\sin(\theta)}$$

Ex. 1 A sleigh is being pulled with a 5N force at an angle of 30° with the ground.

a) Calculate the force that is pulling the sleigh forward.

b) the force that tends to lift the sleigh.



a) $|\vec{F}_x| = |\vec{F}|\cos\theta$

$= 5\cos 30°$

$= \frac{5\sqrt{3}}{2}$

$= 4.33$

∴ The force that is pulling the sleigh forward is ~4.33 N

b) $|\vec{F}_y| = |\vec{F}|\sin\theta$

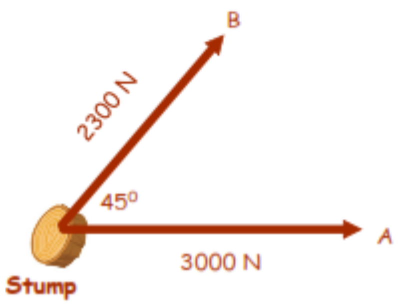
$= 5\sin 30°$

$= 2.5$

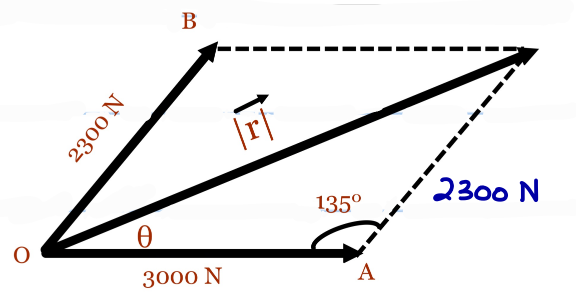
The force that tends to lift the sleigh is 2.5 N

Ex. 2: Two tractors are being used to pull a tree stump out of the ground. The larger tractor pulls with a force of 3000 N[E]. The smaller tractor pulls with a force of 2300 N [NE]. Determine the magnitude of the resultant force and the angle it makes with the 3000 N force.

position diagram



Vector diagram



Let $\vec{r}$ represent the resultant force

Using cosine law

$|\vec{r}|^2 = 3000^2 + 2300^2 - 2(3000)(2300)\cos 135°$

$|\vec{r}|^2 = 24048073\ 58$

$|\vec{r}| = 4903.88352$

$\boxed{|\vec{r}| = 4904}$

Using sine law

$\frac{\sin 135°}{4903.8} = \frac{\sin\theta}{2300}$

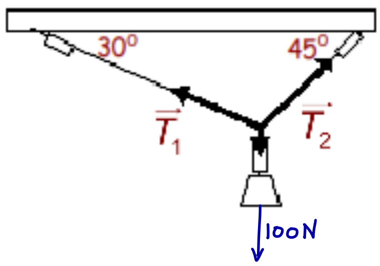
$\theta = \sin^{-1}\left[\frac{2300\sin 135°}{4903.8}\right]$

$\boxed{\theta = 19.4°}$

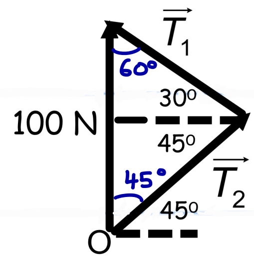
The resultant force is 4904 N [E19 4°N] or [N70.6°E]

Ex. 3: A 100 N weight is suspended from the ceiling by two ropes that make angles of 30° and 45° with the ceiling. Determine the tension in each rope.

position diagram



Vector diagram



Let $\vec{T}_1$ and $\vec{T}_2$ represent the tension in the two ropes

Using sine law

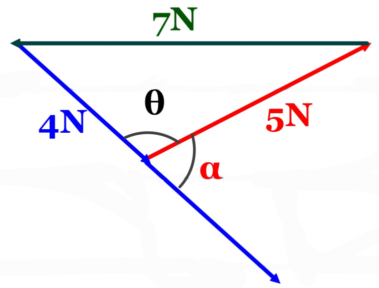
$$\frac{\sin 75^\circ}{100} = \frac{\sin 60^\circ}{|\vec{T}_2|} = \frac{\sin 45^\circ}{|\vec{T}_1|}$$

$$|\vec{T}_2| = \frac{100 \sin 60^\circ}{\sin 75^\circ} \qquad |\vec{T}_1| = \frac{100 \sin 45^\circ}{\sin 75^\circ}$$

$$\boxed{|\vec{T}_2| = 89.7} \qquad \boxed{|\vec{T}_1| = 73.2}$$

The tensions are 89.7 N and 73.2 N in the direction of the ropes

Ex. 4: Three forces having magnitudes of 4 N, 5 N, and 7 N are in a state of equilibrium. Calculate the angle between the two smaller forces.



Using Cosine law

$$7^2 = 4^2 + 5^2 - 2(4)(5)\cos\theta$$

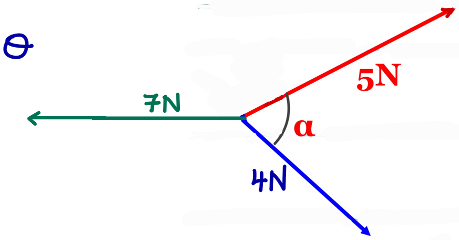
$$\theta = \cos^{-1}\left(\frac{4^2+5^2-7^2}{2(4)(5)}\right)$$

$$\theta = 101.53°$$

$$\theta = 101.5°$$

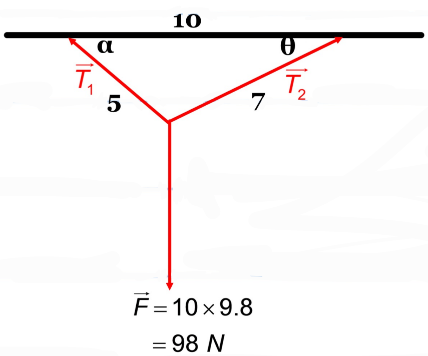
$$\alpha = 180° - 101.5°$$

$$\boxed{\alpha = 78.5°}$$

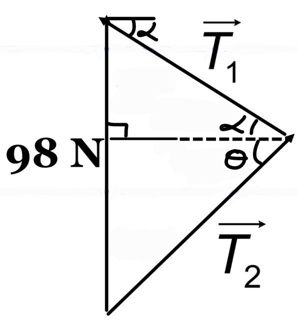


The angle between the 2 smaller forces is 78.5°

Ex. 5: A 10 kg mass is supported by two strings of length 5 m and 7 m attached to two points in the ceiling 10 m apart. Find the tension in each string.



Note
Force = mass x acceleration due to gravity



Let the tensions be $\vec{T}_1$ and $\vec{T}_2$

Using Cosine law

$$5^2 = 10^2 + 7^2 - 2(10)(7)\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{10^2+7^2-5^2}{2(10)(7)}\right)$$

$$\theta = 27.7°$$

$$7^2 = 5^2 + 10^2 - 2(5)(10)\cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{5^2+10^2-7^2}{2(5)(10)}\right)$$

$$\alpha = 40.5°$$

Using Sine law

$$\frac{\sin 68.2°}{98} = \frac{\sin(90° - 27.7°)}{|\vec{T}_1|} = \frac{\sin(90° - 40.5°)}{|\vec{T}_2|}$$

$$|\vec{T}_1| = \frac{98\sin 62.3°}{\sin 68.2°} \qquad |\vec{T}_2| = \frac{98\sin 49.5°}{\sin 68.2°}$$

$$\boxed{|\vec{T}_1| = 93.5} \qquad \boxed{|\vec{T}_2| = 80.3}$$

The tensions are 93.5 N and 80.3 N in the direction of the strings

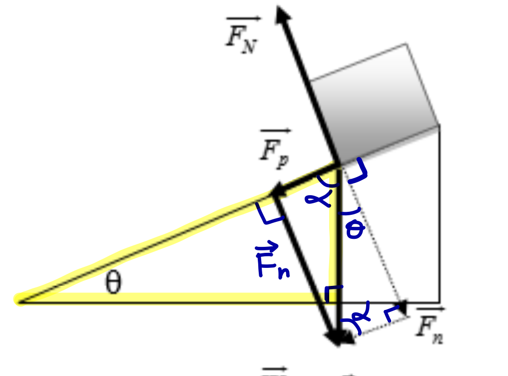
A Ramp Problem

The next example shows that rectangular components do not necessarily have to be horizontal or vertical.

A box weighing $\vec{W}$ Newton is resting on a ramp that is inclined at an angle of θ°. Resolve the weight into the rectangular components, $\vec{F_p}$, the force parallel to the surface , and $\vec{F_n}$, the force perpendicular to the surface. Note $\vec{F_N}$ is the force of the ramp pushing against the box. This force counteracts the component of gravity in the opposite direction to keep the box at rest.

$|\vec{F_n}| = m|\vec{g}|\cos\theta$

$|\vec{F_p}| = m|\vec{g}|\sin\theta$

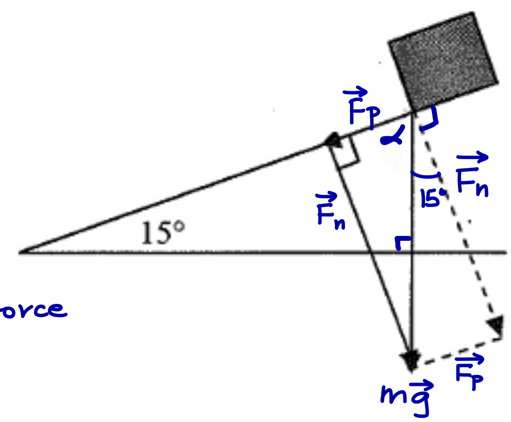


$|\vec{g}| = 9.8 m/s^2$

Ex. 7: Components of the forces of gravity A 20-kg trunk is resting on a ramp inclined at an angle of 15°. Calculate the components of the force of gravity on the trunk that are parallel and perpendicular to the ramp.

$|\vec{F_p}| = m|\vec{g}|\sin\theta$
$= 20(9.8)\sin 15°$
$\doteq 50.7$

$|\vec{F_n}| = m|\vec{g}|\cos\theta$
$= 20(9.8)\cos 15°$
$= 189$



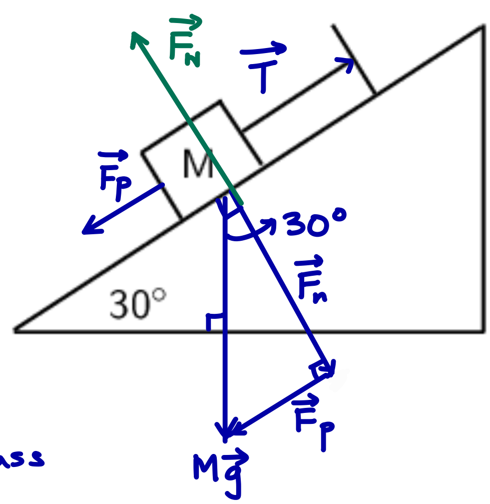
The force parallel to the ramp is 50.7 N and the force perpendicular to the ramp is 189 N

Ex.8 : A block of mass M is held stationary by a rope of negligible mass. The block rests on a frictionless plane which is inclined at 30° to the horizontal.

a) Resolve the force due to gravity into components that are parallel and perpendicular to the plane.

Force parallel to the plane ⇒ $|\vec{F_p}| = M|\vec{g}|\sin 30°$

Force perpendicular to the plane ⇒ $|\vec{F_n}| = M|\vec{g}|\cos 30°$



b) Calculate the tension in the rope .($\vec{g} = 9.8$ m/s^2)

$\vec{F_p} + \vec{T} = \vec{0}$

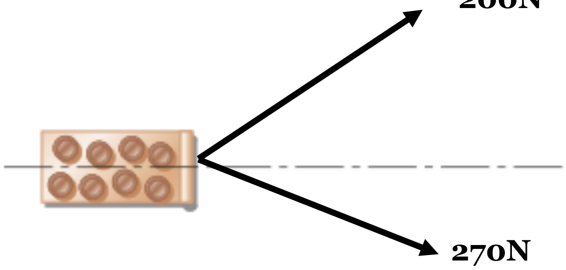
$\vec{F_p} = -\vec{T}$

$|\vec{T}| = M(9.8)\sin 30°$ where M is the mass of the box
$= 4.9M$

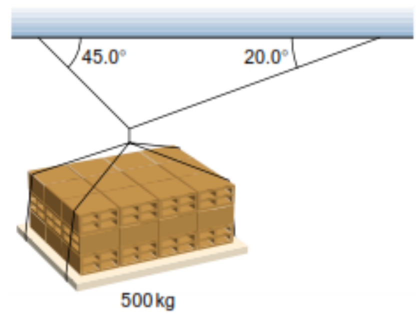
The tension in the rope is 4.9M N

Practice Questions

1. Two horses pull a load. The ropes between the horses and the load are at an angle of 80° to each other. One horse pulls with a force of 200 N (newton), and the other with a force of 270 N. Here is a diagram to illustrate the two forces. Calculate the resultant force.[Ans.363 N at 47° to 200N]

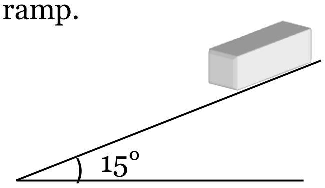


2. A mass of 500 kg is supported by two cables as illustrated. What is the tension in each cable? ($\vec{g} = 9.8 m/s^2$)
[Ans. 3823 N and 5080.5N]

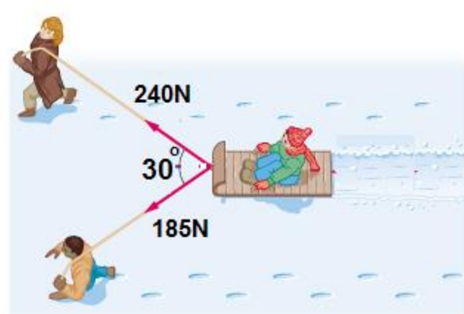


3. A lawnmower is pushed across a lawn by applying a force of 95 N along the handle of the mower. The handle makes an angle of 60.0° with the horizontal.
   a. What are the horizontal and vertical components of the force?[Ans. F_x = 48N ,F_y = 82N]
   b. The handle is lowered so that it makes an angle of 30.0° with the horizontal. What are the horizontal and vertical components of the force? ?[Ans. F_x = 82N ,F_y = 48N]

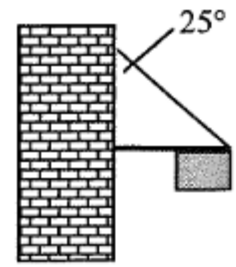
4. 20-kg trunk is resting on a ramp inclined at an angle of 15°. Calculate the components of force of gravity on the trunk that are parallel and perpendicular to the ramp.
[Ans. $\vec{F}_p = 50.7$ N ,$\vec{F}_n = 189.3$N]



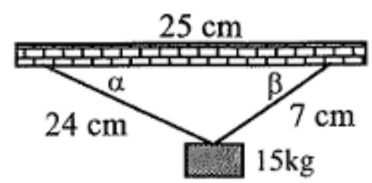
5. Using two ropes that make an angle of 30° to each other, Jack and Alex pull Bill in a sleigh. Jack pulls with 240N force and Alex pulls with a force of 185N. Determine the magnitude and direction of the equilibrant force. [Ans. 410.8 N , makes an angle of 167° with the larger force]



6. An advertising sign is supported by a horizontal steel brace extending at right angles from the side of a building, and by a wire attached to the building above the brace at an angle of 25° .If the force of gravity on the sign is 850 N, find the tension in the wire and the compression in the steel brace.

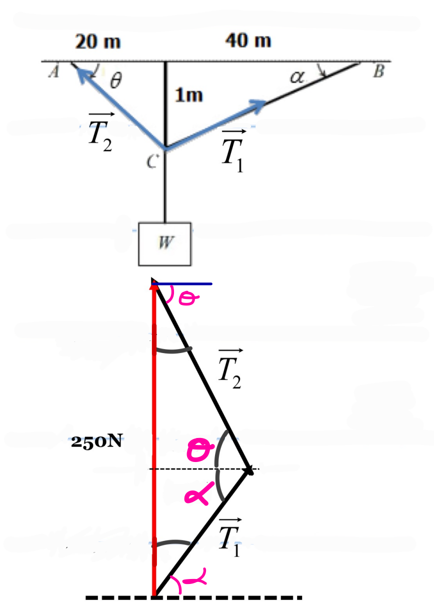


7. An object of 15 kg is suspended by two cords of lengths 7 cm and 24 cm, and these two cords are 25 cm apart. Find the tension in each cord. [Ans. The tensions of two cords are 141N and 41.3N]



Warm Up

A ski chairlift is suspended between two towers that are 60 m apart horizontally. When the chairlift is 20m from one tower, the cable sags 1m.The chairlift is loaded with four skiers with a combined weight of 250N (including the mass of the chair). What are the tensions on the two parts of the cable?



Let the diagram be as labelled

$\tan\theta = \frac{1}{20}$ $\qquad$ $\tan\alpha = \frac{1}{40}$

$\theta = \tan^{-1}\left(\frac{1}{20}\right)$ $\qquad$ $\alpha = \tan^{-1}\left(\frac{1}{40}\right)$

$\theta = 2.86°$ $\qquad$ $\alpha = 1.43°$

Using sine law,

$$\frac{\sin 4.29°}{250} = \frac{\sin 87.14°}{|\vec{T}_1|} = \frac{\sin 88.57°}{|\vec{T}_2|}$$

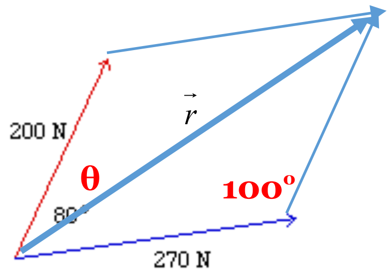
$|\vec{T}_1| = \frac{250 \sin 87.14°}{\sin 4.29°}$ $\qquad$ $|\vec{T}_2| = \frac{250 \sin 88.57°}{\sin 4.29°}$

$\boxed{|\vec{T}_1| \doteq 3338}$ $\qquad$ $\boxed{|\vec{T}_2| \doteq 3341}$

∴ The tensions on the two parts of the cable are 3338 N and 3341 N in the direction of the cables

Practice Questions-Solutions

1. Two horses pull a load. The ropes between the horses and the load are at an angle of 80° to each other. One horse pulls with a force of 200 N (newton), and the other with a force of 270 N. Here is a diagram to illustrate the two forces. Calculate the resultant force.



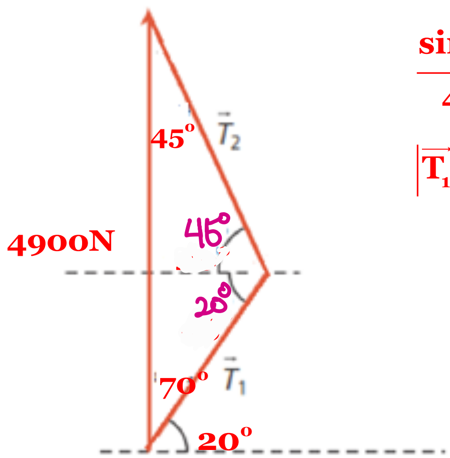
$$|\vec{r}|^2 = 200^2 + 270^2 - 2(200)(270)\cos(100^\circ)$$

$$|\vec{r}| \doteq 363\,\text{N}$$

$$\frac{\sin(\theta)}{270} = \frac{\sin(100^\circ)}{363}$$

$$\theta \doteq 47^\circ$$

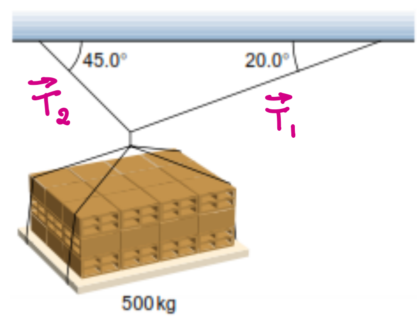
2. A massof 500 kg is supported by two cables as illustrated. What is the tension in each cable?



Vector Diagram

$$\frac{\sin(65^\circ)}{4900} = \frac{\sin(45^\circ)}{|\vec{T}_1|} = \frac{\sin(70^\circ)}{|\vec{T}_2|}$$

$$|\vec{T}_1| = 3823\text{N}\ ,\ |\vec{T}_2| = 5080.5\ \text{N}$$



3. A lawnmower is pushed across a lawn by applying a force of 95 N along the handle of the mower. The handle makes an angle of 60.0◦ with the horizontal.

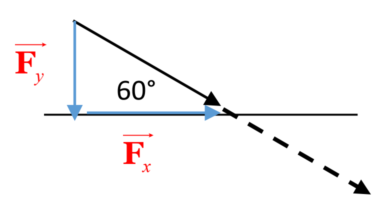
   a. What are the horizontal and vertical components of the force?

   b. The handle is lowered so that it makes an angle of 30.0◦ with the horizontal. What are the horizontal and vertical components of the force? ?

a)

$$|\vec{F}_x| = mg\cos(\theta)$$
$$= 95(9.8)\cos(60^\circ)$$
$$\doteq 48\text{N}$$

$$|\vec{F}_y| = mg\sin(\theta)$$
$$= 95(9.8)\sin(60^\circ)$$
$$\doteq 82\,\text{N}$$

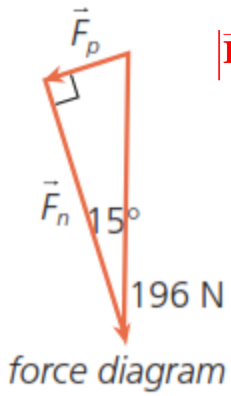


b)

$$|\vec{F}_x| = mg\cos(\theta)$$
$$= 95(9.8)\cos(30^\circ)$$
$$\doteq 82\text{N}$$

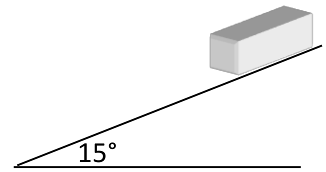
$$|\vec{F}_y| = mg\sin(\theta)$$
$$= 95(9.8)\sin(30^\circ)$$
$$\doteq 48\,\text{N}$$

4. A 20-kg trunk is resting on a ramp inclined at an angle of 15°. Calculate the components of force of gravity on the trunk that are parallel and perpendicular to the ramp. Describe the physical consequences of each.



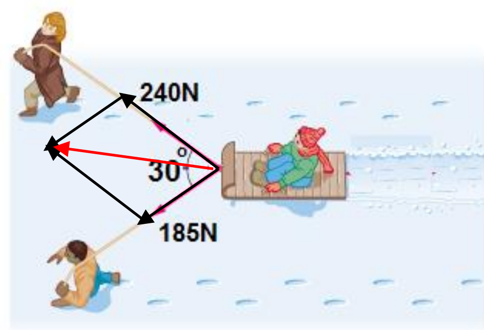
$$|\vec{\mathbf{F}}_n| = |\vec{\mathbf{F}}_g|\cos(\theta)$$
$$= 90 \times 9.8 \times \cos(15^\circ)$$
$$\doteq 189\,\mathbf{N}$$

$$|\vec{\mathbf{F}}_p| = |\vec{\mathbf{F}}_g|\sin(\theta)$$
$$= 90 \times 9.8 \times \sin(15^\circ)$$
$$\doteq 51\,\mathbf{N}$$



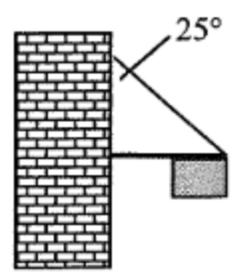
The parallel component points down the slope of the ramp. It tends to cause the trunk to slide down the slope. It is opposed by the force of friction acting up the slope. The perpendicular component presses the trunk against the ramp. The magnitude of the force of friction is proportional to this component.

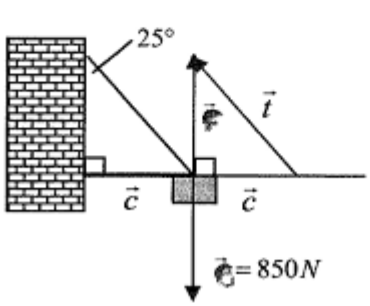
5. Using two ropes that make an angle of 30° to each other, Jack and Alex pull Bill in a sleigh. Jack pulls with 240N force and Alex pulls with a force of 185N. Determine the magnitude and direction of the equilibrant force.



$$|\vec{\mathbf{R}}| = \sqrt{240^2 + 185^2 - 2(240)(185)\cos 150^\circ}$$
$$= 410.8\mathrm{N}$$
$$\frac{\sin 150^\circ}{410.8} = \frac{\sin\theta}{185}$$
$$\angle\theta = 13^\circ$$
$$\therefore \vec{\mathbf{E}} = 410.8\mathrm{N} \text{ makes an angle of } 180^\circ - 13^\circ = 167^\circ \text{ with the larger force}$$

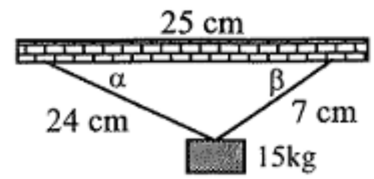
6. An advertising sign is supported by a horizontal steel brace extending at right angles from the side of a building, and by a wire attached to the building above the brace at an angle of 25°. If the force of gravity on the sign is 850 N, find the tension in the wire and the compression in the steel brace. [Ans. The tension on the wire is 937.9N and the compression in the steel brace is 396.4 N]

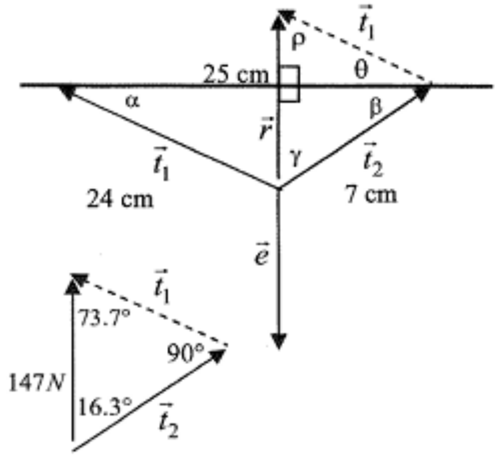




$$\cos 25^\circ = \frac{850}{|\vec{t}|}$$
$$|\vec{t}| \approx 937.9N$$
$$\tan 25^\circ = \frac{|\vec{c}|}{850}$$
$$|\vec{c}| \approx 396.4N$$

7. An object of 15 kg is suspended by two cords of lengths 7 cm and 24 cm, and these two cords are 25 cm apart. Find the tension in each cord.[Ans. The tensions of two cords are 141N and 41.3N]





Let $\vec{t}_1$ and $\vec{t}_2$ represent the tensions in both cords.

$|\vec{e}| = |\vec{r}| = 15kg \times 9.8N/kg = 147N$

$$\cos\alpha = \frac{24^2 + 25^2 - 7^2}{2(25)(24)}$$

$\alpha \approx 16.3°$

$$\cos\beta = \frac{7^2 + 25^2 - 24^2}{2(7)(25)}$$

$\beta \approx 73.7°$

$\theta = \alpha \approx 16.3°$ (Alt ∠)

$\gamma \approx 16.3°$ (Supp ∠)

$\rho \approx 73.7°$ (Supp ∠)

$$\frac{\sin 16.3°}{|\vec{t}_1|} \approx \frac{\sin 90°}{147}$$

$|\vec{t}_1| \approx 41.3N$

$$\frac{\sin 73.7°}{|\vec{t}_2|} \approx \frac{\sin 90°}{147}$$

$|\vec{t}_2| \approx 141N$

∴ The tensions of two cords are 141N and 41.3N.

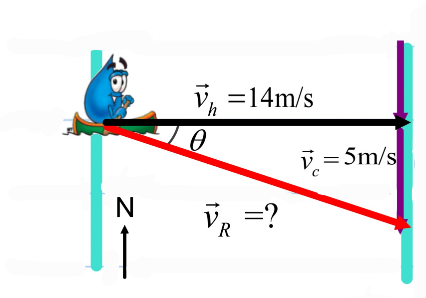
Velocity as a Vector

Velocity measures the direction and the rate of change in the position of an object.

- Velocity is a vector because it has both ___magnitude___ and ___direction___.
- **Air speed (water speed)** is the speed of a plane (boat) relative to a person on board.
- **Ground speed** is the speed of a plane (boat) relative to a person **on the ground** and includes the effect of wind (current).

Ex1: A boat with a forward velocity of 14 m/s is traveling across a river, directly towards the opposite shore. At the same time, a current of 5 m/s carries the boat down the river.

(a) Determine the resultant velocity of the boat.



$\vec{V}_h$ = velocity of the boat with respect to water
= 14 m/s [E]

$\vec{V}_c$ = velocity of water with respect to the current
= 5 m/s [S]

$\vec{V}_R$ = velocity of boat with respect to current

$|\vec{V}_R| = \sqrt{14^2 + 5^2}$

$\boxed{|\vec{V}_R| = 14\ 9}$

$\tan\theta = \frac{5}{14}$

$\theta = \tan^{-1}\left(\frac{5}{14}\right)$

$\boxed{\theta = 19\ 7}$

The resultant velocity of the boat is 14 9 m/s [S 70 3° E]

(b) Suppose the river was 100 m across, how long would it take for the boat to cross the river?

$V = \frac{d}{t}$

$t = \frac{d}{V}$

$t = \frac{100}{14}$

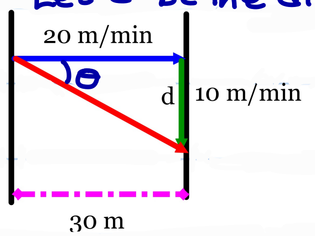
$\boxed{t = 7\ 14}$

It will take 7 14 s to cross the river

Ex2: Alex wishes to use a canoe to cross to the other side of a river, which is 30 m wide. The river is flowing at 10 m/min and Alex can paddle at 20 m/min.

(a) If he points his canoe directly across the river (perpendicular to the bank), where will he land? How long will the crossing take?

Let d be the distance



$t = \frac{d}{v}$

$t = \frac{30}{20}$

$\boxed{t = 1.5}$

$d = vt$

$d = 10(1.5)$

$\boxed{d = 15}$

Method ②

$\tan\theta = \frac{10}{20}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\theta = 26.6°$

$\tan 26.6° = \frac{d}{30}$

$d = 30\left[\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)\right]$

$d = 15$

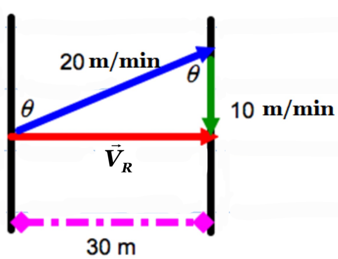
$t = \frac{d}{v}$

$= \frac{15}{10}$

$= 1.5$

Alex will land 15 m downstream from the point directly across from his starting position, taking 1.5 min to do so

(b) In what direction should he aim the canoe in order to land at a point directly opposite his starting point? How long will it take to make this crossing?



Let the diagram be as labelled

$20^2 = 10^2 + |\vec{V}_R|^2$

$|\vec{V}_R| = \sqrt{400 - 100}$

$|\vec{V}_R| = 10\sqrt{3}$

$\sin\theta = \frac{10\sqrt{3}}{20}$

$\boxed{\theta = 60°}$

$t = \frac{d}{v}$

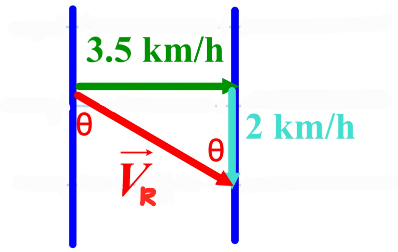
$t = \frac{30}{10\sqrt{3}}$

$t = \sqrt{3}$

$\boxed{t = 1.73}$

Alex should head at 60° to the bank in the direction he is heading. It will take him 1.73 min

Ex3 : **(a)** Fiona heads straight out across a stream flowing at 2 km/hr. She can row at 3.5 km/hr in still water. Determine her resultant velocity.



Let $\vec{V}_R$ be the resultant velocity

$$|\vec{V}_R|^2 = 3.5^2 + 2^2$$

$$|\vec{V}_R| = \sqrt{16.25}$$

$$\boxed{|\vec{V}_R| = 4.03}$$

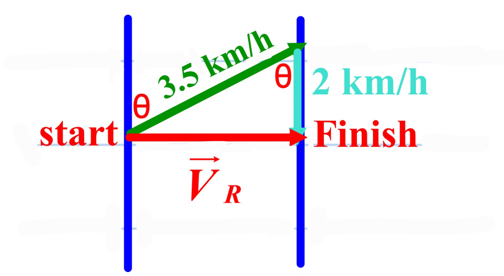
$$\tan\theta = \frac{3.5}{2}$$

$$\theta = \tan^{-1}\left(\frac{3.5}{2}\right)$$

$$\boxed{\theta = 60.3^\circ}$$

∴ Fiona's resultant velocity is 4.03 km/h at 60.3° to the bank

(b) Suppose Fiona needs to land on the bank directly opposite her starting position. Which direction would she have to steer and what would be her resultant velocity?



Let the diagram be as labelled

$$\cos\theta = \frac{2}{3.5}$$

$$\theta = \cos^{-1}\left(\frac{2}{3.5}\right)$$

$$\boxed{\theta = 55.2^\circ}$$

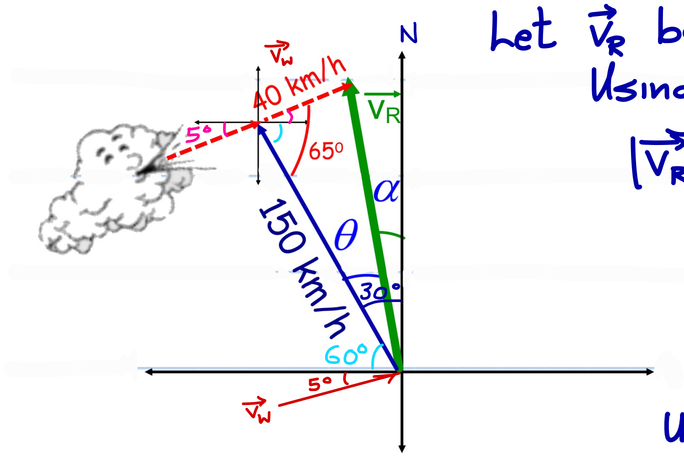
$$|\vec{V}_R|^2 = 3.5^2 - 2^2$$

$$|\vec{V}_R| = \sqrt{8.25}$$

$$\boxed{|\vec{V}_R| = 2.87}$$

Fiona must head 55.2° to the bank. Her resultant velocity is 2.87 km/h

Ex.4 : A small aircraft is flying on a heading [N 30° W] at a constant speed of 150 km/h. The wind is blowing **from** 5° south of west with a speed of 40 km/h. Determine the actual speed and direction of the aircraft relative to the ground



Let $\vec{V}_R$ be the resultant velocity

Using Cosine law

$$|\vec{V}_R|^2 = 40^2 + 150^2 - 2(40)(150)\cos 65^\circ$$

$$|\vec{V}_R| = 137.944$$

$$\boxed{|\vec{V}_R| = 138}$$

Using sine law,

$$\frac{\sin 65^\circ}{137.944} = \frac{\sin\theta}{40}$$

$$\theta = \sin^{-1}\left(\frac{40\sin 65^\circ}{137.944}\right)$$

$$\theta = 15.2^\circ$$

$$\alpha = 30^\circ - 15.2^\circ$$

$$\boxed{\alpha = 14.8^\circ}$$

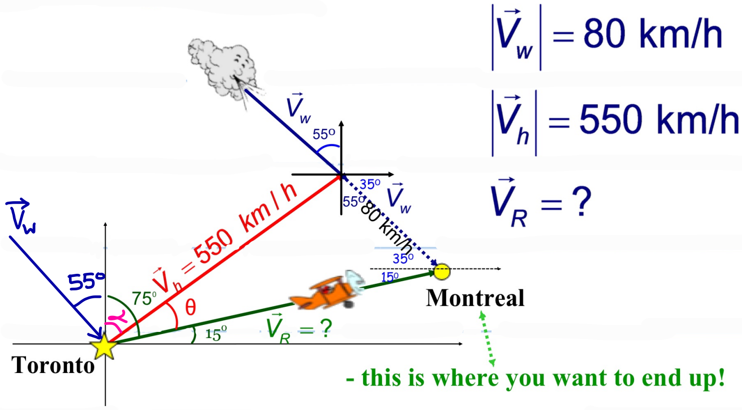
The ground speed of the aircraft is 138 km/h and the direction is N 14.8° W

Ex5 : A pilot wishes to fly from Toronto to Montreal a distance of 500 km on a heading of [N 75° E]. The airspeed of the plane is 550 km/h. An 80 km/h wind is blowing from [N 55° W].

a) What heading should the pilot take to reach his destination?

b) What will be the speed of the plane relative to the ground?(groundspeed)

c) How long will the trip take?



$|\vec{V}_w| = 80$ km/h

$|\vec{V}_h| = 550$ km/h

$\vec{V}_R = ?$

a) Using sine law,

$$\frac{\sin 50^\circ}{550} = \frac{\sin\theta}{80}$$

$$\theta = \sin^{-1}\left(\frac{80\sin 50^\circ}{550}\right)$$

$$\theta = 6\ 39^\circ$$

$$\alpha = 75^\circ - 6\ 39^\circ$$

$$\boxed{\alpha = 68\ 6^\circ}$$

The pilot should head in the direction of N68 6°E

b) Using cosine law,

$$|\vec{V}_R|^2 = 550^2 + 80^2 - 2(550)(80)\cos(55^\circ + 68\ 6^\circ)$$

$$\boxed{|\vec{V}_R| = 598}$$

The ground speed is 598 km/h

c) $t = \frac{d}{s}$

$$t = \frac{500}{597\ 9}$$

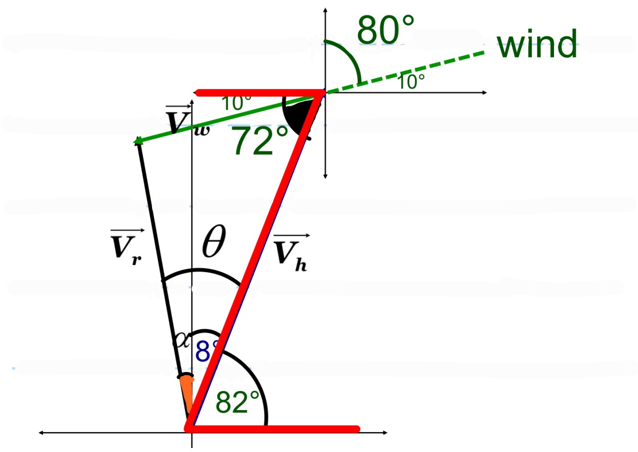
$$t = 0\ 836 \text{ h}$$

$$\boxed{t = 50\ 2} \text{ min}$$

The trip will take 50 2 min

Exit Card!

A light plane is travelling at 175 km/h on a heading of [N8°E] encounters a wind of 40 km/hr from [N80°E]. Determine the plane's ground velocity.



Let the diagram be as labelled

$|\vec{V}_r|^2 = 40^2 + 175^2 - 2(40)(175)\cos 72^\circ$

$\boxed{|\vec{V}_r| = 167}$

$\frac{\sin\theta}{40} = \frac{\sin 72^\circ}{167}$

$\theta = \sin^{-1}\left(\frac{40\sin 72^\circ}{167}\right)$

$\theta = 13.2^\circ$

$\alpha = 13.2^\circ - 8^\circ$

$\boxed{\alpha = 5.2^\circ}$

. The ground velocity is ~167 km/h in the direction of N 5.2°W

Practice on Velocity as Vectors

1. Chris is hoping to catch a connecting flight, and time is short. He must cover a distance of 800 m to his next gate in 6 min . Fortunately, a moving walkway extends from gate to gate, going 40 m/min .
   a) If the walkway is going in the same direction as he is, how fast must he walk on it to make his flight?
   b) If he gets on the walkway and then realizes it is going in the opposite direction, how fast must he walk on it to get to the gate on time?

2. Thieves are fleeing in a stolen boat travelling at 30 km/h due west. A police boat is sent to catch them. When the stolen boat is 3 km due north of the police, the police set out at a speed of
   40 km/h .
   a) In what direction must the police head in order to intercept the thieves?
   b) When will the interception occur?

3. An airplane which flies at 200 km/h is headed due north. A wind is blowing due east at 40 km/h.
   a) What is the magnitude and direction of the plane's velocity relative to the ground?
   b) After flying for 90 minutes under these conditions, what is the location of the plane?

4. An airplane is flying at 150 km/h at a heading of W 10° N. When it lands 2 hours later, its location is 275 km from the starting point, at a heading of W 20° N. What is the magnitude and direction of the wind velocity?

5. A pilot is planning his flight to an airport which is 400 km southeast of his starting location. His plane flies at 250 km/h but a wind of 20 km/h is blowing from the southwest.
   a) What heading should he choose for the plane?
   b) How long will the journey take?

6. Alex wishes to use a canoe to cross to the other side of a river which is 30 m wide. The river is flowing at 10 m/min and Alex can paddle at 20 m/min. Her goal is a dock which is 6 m downstream from a point directly opposite her starting point. In what direction should she aim her canoe? How long will it take to make this crossing?

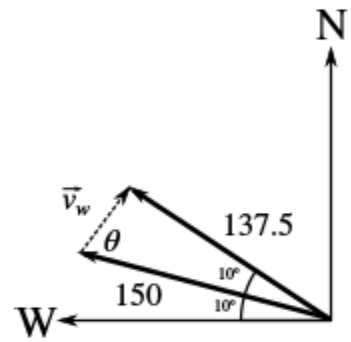
Answers

1. a) 93.3 m/min — b) 173.3 m/min
2. a) [N 48.6° W] — b) 6.8 minutes
3. a) 203.9 km/h [N 11.3° E] — b) 305.9 km [N 11.3° E] from starting point.
4. 28.0 km/h [E 48.5° N]
5. a) [E 49.6° S] — b) 1 hour, 36 minutes
6. 72° upstream from the bank, 1.6 min

Velocity as a Vector-Partial Solutions

1. There is no solution provided for this question.

2. a)



$$\sin(\theta) = \frac{30t}{40t}$$
$$= \frac{3}{4}$$
$$\theta = 48.6°$$

The police should head in a direction [N 48.6° W].

b)

$$\tan(\theta) = \frac{30t}{3}$$
$$\tan(48.6°) = 10t$$
$$t \doteq 0.113h \text{ or } 6.8 \text{ min}$$

1. **The interception will occur after 6.8 minutes.**

3. There is no solution provided for this question.

4. Let θ be the angle formed between the wind velocity $\vec{v_w}$ and the aircraft's velocity. Let $\vec{v}$ be the resultant velocity. Then $|\vec{v}| = \frac{275}{2} = 137.5 \text{ km/h}$, with a direction of [W 20° N], as shown in the diagram.

Applying the cosine law,

$$|\vec{v_w}|^2 = 150^2 + 137.5^2 - 2(150)(137.5)\cos(10°)$$
$$|\vec{v_w}| \doteq 28 \text{ m/s}$$

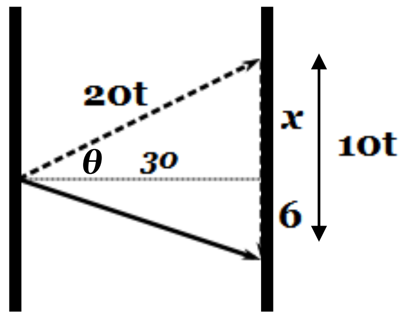
Using the sine law,

$$\frac{\sin(\theta)}{137.5} = \frac{\sin(10°)}{28}$$
$$\sin(\theta) = \frac{137.5\sin(10°)}{28}$$
$$\theta \doteq 58.5°$$

Therefore, the wind velocity is 28 m/s [E 48.5° N].

5. There is no solution provided for this question.

6.



Let t be the time for the entire crossing. Let θ be the angle that her boat makes with the perpendicular to the current when it is launched.

$$\sin(\theta)=\frac{x}{20t}\rightarrow x=20t\sin(\theta)$$

$$x+6=10t\rightarrow 6=10t-x$$

$$6=10t-20t\sin(\theta)\qquad(1)$$

$$\cos\theta=\frac{30}{20t}\rightarrow 30=20t\cos(\theta)\qquad(2)$$

$$5\times(1):\ 30=50t-100t\sin(\theta)\qquad(3)$$

$$\text{sub. (2) into (3)}:\ 20t\cos(\theta)=50t-100t\sin(\theta)$$

$$2\cos(\theta)=5-10\sin(\theta)$$

raise both sides to the power of two:

$$4\cos^2(\theta)=25-100\sin(\theta)+100\sin^2(\theta)$$

$$4\left(1-\sin^2(\theta)\right)=25-100\sin(\theta)+100\sin^2(\theta)$$

$$104\sin^2(\theta)-100\sin(\theta)+21=0$$

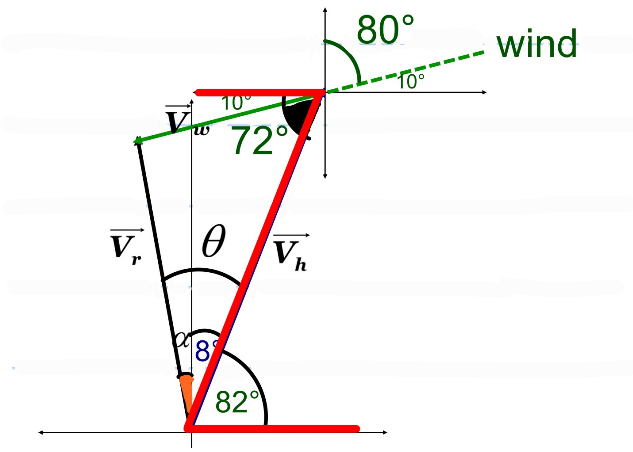
$$\sin(\theta)=0.3098 \text{ and } \theta\approx 18^{\circ}.$$

$$\alpha=90^{\circ}-18^{\circ}=72^{\circ}$$

$$30=20t\cos(\theta)\rightarrow t=\frac{3}{2\cos(18^{\circ})}\approx 1.6 \text{ minutes.}$$

Exit Card!

A light plane is travelling at 175 km/h on a heading of [N8°E] encounters a wind of 40 km/hr from [N80°E]. Determine the plane's ground velocity.



Let the diagram be as labelled

$$|\vec{V}_r|^2 = 40^2 + 175^2 - 2(40)(175)\cos 72^\circ$$

$$\boxed{|\vec{V}_r| = 167}$$

$$\frac{\sin\theta}{40} = \frac{\sin 72^\circ}{167}$$

$$\theta = \sin^{-1}\left(\frac{40\sin 72^\circ}{167}\right)$$

$$\theta = 13.2^\circ$$

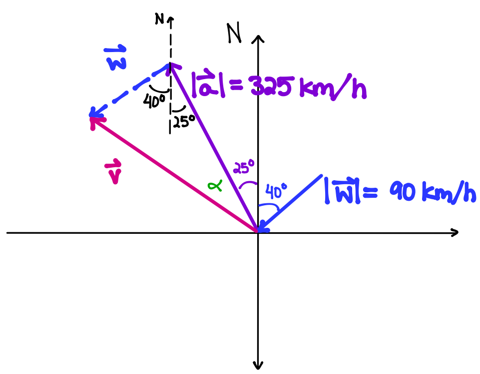
$$\alpha = 13.2^\circ - 8^\circ$$

$$\boxed{\alpha = 5.2^\circ}$$

. The ground velocity is ~167 km/h in the direction of N 5.2°W

Warm-Up

1. A plane is steering [N25°W] at an airspeed of 325 km/hr. The wind is from [N40°E] at 90 km/hr. Find the ground speed of the plane and its course, [298 km/h [N40.9W]]



Let $|\vec{v}|$ be the ground speed.

$|\vec{v}|^2 = (90)^2 + (325)^2 - 2(90)(325)\cos 65°$

$|\vec{v}| \doteq 298.3317477$

$\frac{\sin \alpha}{90} = \frac{\sin 65°}{298.3317477}$

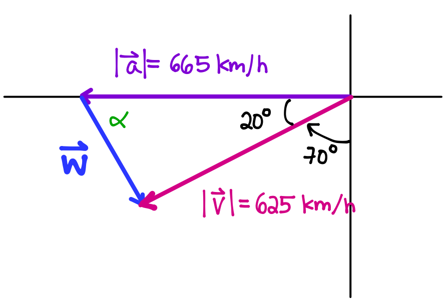
$\alpha \doteq 15.9°$

Direction $\doteq 15.9° + 25°$

$\doteq 40.9°$

∴ The ground speed is approx. 298 km/h in a direction of [N 40.9°W].

2. A plane is heading [S70°W] with a ground speed of 625 km/hr. If the pilot is steering west at an airspeed of 665km/hr, what must be the wind speed and wind direction. [227 km/h [S20°E]]



$|\vec{w}|^2 = (665)^2 + (625)^2 - 2(665)(625)\cos 20°$

$|\vec{w}| = 227.4434139$

$\frac{\sin \alpha}{625} = \frac{\sin 20°}{227.4434139}$

$\alpha \doteq 70.1°$

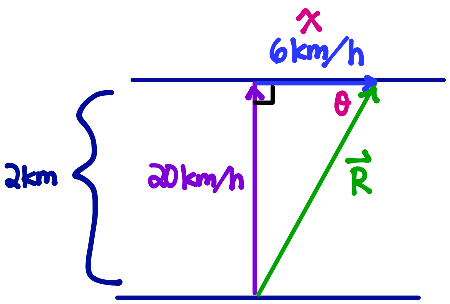
Direction $= 90° - 70.1°$

$\doteq 19.9°$

∴ The wind speed is approx. 227 km/h in the direction of [S 19.9° E]

3. A river is 2 km wide and flows at 6 km/hr. A motor boat that has a speed of 20 km/hr in still water heads out from one bank perpendicular to the current. A marina lies directly across the river on the opposite bank.

   a) How far downstream f ram the marina will the boat reach the other bank? [0.6 km downstream]
   b) How long will it take? [6 minutes]



a) Let x be the distance downstream

$\tan\theta = \frac{20}{6}$ ← speeds

$\theta \doteq 73.3°$

$\tan\theta = \frac{2}{x}$ ← distance

$x = \frac{2}{\tan 73.3°}$

$x = 0.6$

∴ The boat will be approx. 0.6 km downstream.

b) Let $\vec{R}$ be the resultant vector

distance: $|\vec{R}|^2 = (2)^2 + (0.6)^2$

$|\vec{R}| \doteq 2.09$

speed: $|\vec{R}|^2 = (20)^2 + (6)^2$

$|\vec{R}| = 2\sqrt{109}$

$\doteq 20.9$

$t \doteq \frac{2.09}{20.9}$

$= 0.1\text{ h}$

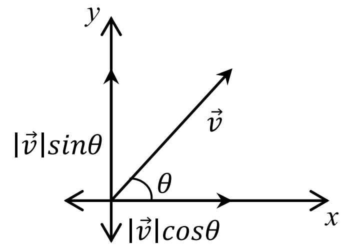
$= 6\text{ min}$

∴ It will take 6 min

CARTESIAN VECTORS

A vector can be identified as a Cartesian Vector if its endpoints can be defined using Cartesian Coordinates.

To write a geometric vector $\vec{v}$ in Cartesian form, you need to use trigonometry. The magnitude of the horizontal component is $|\vec{v}|cos\theta$, and the magnitude of the vertical component is $|\vec{v}|sin\theta$, where θ is the angle $\vec{v}$ makes with the horizontal, or the positive x-axis. Thus, $\vec{v} = [|\vec{v}|cos\theta, |\vec{v}|sin\theta]$.



Note: $\theta = \tan^{-1}\left(\frac{|\vec{v}|\sin\theta}{|\vec{v}|\cos\theta}\right)$

Example 1. Write a force of 300N at 30° to the horizontal in Cartesian form.

Let $\vec{F}$ be the force

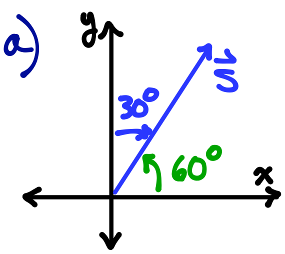
$$\vec{F} = (|\vec{F}|\cos\theta, |\vec{F}|\sin\theta)$$
$$= (300\cos 30°, 300\sin 30°)$$
$$= (150\sqrt{3}, 150)$$

∴ The force in Cartesian form is $(150\sqrt{3}, 150)$.

Example 2. A ship's course is set to travel at 45 km/h, relative to the water, on a heading of 030°. A current of 10 km/h is flowing from a bearing of 140°.

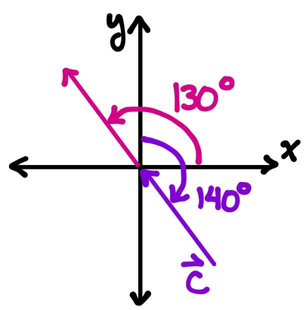
a) Write each vector as a Cartesian vector.
b) Determine the resultant velocity of the ship.

a)



Let $\vec{s}$ be the velocity of the ship.

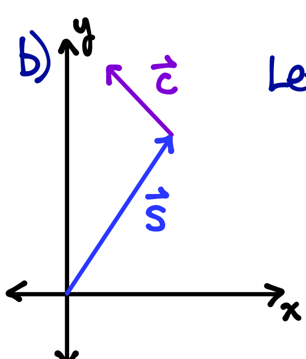
$$\vec{s} = (45\cos 60°, 45\sin 60°)$$
$$= \left(\frac{45}{2}, \frac{45\sqrt{3}}{2}\right)$$



Let $\vec{c}$ be the velocity of the current

$$\vec{c} = (10\cos 130°, 10\sin 130°)$$
$$= (-6.43, 7.66)$$

b)



Let $\vec{R}$ be the resultant velocity

$$\vec{R} = \vec{s} + \vec{c}$$
$$= (45\cos 60°, 45\sin 60°) + (10\cos 130°, 10\sin 130°)$$
$$= (16.1, 46.6)$$

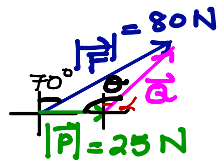
$$|\vec{R}| = \sqrt{(16.1)^2 + (46.6)^2}$$
$$\doteq 49.3$$

$$\text{Bearing} = \tan^{-1}\left(\frac{16.1}{46.6}\right)$$
$$= 019.1°$$

∴ The ship is travelling at 49.3 km/h at a bearing of 019.1°.

Velocity, Forces, and Cross Product

1. The resultant of $\vec{P}$ and $\vec{Q}$ is a force, $\vec{F}$ is $80N$ $[N70°E]$ and $\vec{P}$ is $25N$ $[E]$. Find the magnitude and direction of $\vec{Q}$ $[57.15N\ [N61.1°E]]$



Let the diagram be as labelled
Using cosine law

$|\vec{Q}|^2 = 80^2 + 25^2 - 2(80)(25)\cos 20°$

$\vec{Q} = 57.2$

Find angle θ $\quad 80^2 = 25^2 + 57.2^2 - 2(25)(57.2)\cos\theta$

Note If you use sine law you must find angle β first and then find θ to avoid ambiguous case

$\theta = \cos^{-1}\left[\dfrac{80^2 - 25^2 - 57.2^2}{-2(25)(57.2)}\right]$

$\theta = 151.1°$

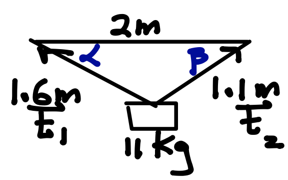
$\alpha = 151.1° - 90°$

$\alpha = 61.1°$

magnitude is 57.2N
direction is N61.1°E

The longest side must be opposite the largest angle

2. A particle of mass 11 kg is suspended from a horizontal ceiling by cords from two points A and B on a horizontal ceiling such that $AB = 2\ m$. The length of the cords are 1.6 m and 1.1 m. Calculate the tension in each cord. $[90.31N\ and\ 65.1N]$



Using cosine law,

$1.1^2 = 1.6^2 + 2^2 - 2(1.6)(2)\cos\alpha$

$\cos\alpha = \dfrac{1.1^2 - 1.6^2 - 2^2}{-2(1.6)(2)}$

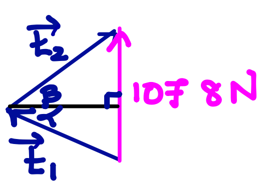
$\alpha = 33.3°$

$1.6^2 = 1.1^2 + 2^2 - 2(1.1)(2)\cos\beta$

$\beta = 53.0°$

Force $= 11kg \times 9.8 m/s^2$

$= 107.8\ N$



Using sine law,

$\dfrac{\sin 86.3°}{107.8} = \dfrac{\sin 37°}{|\vec{T_1}|}$

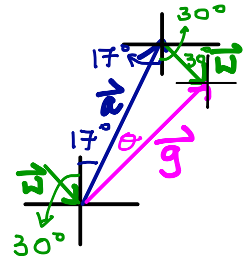
$|\vec{T_1}| = 65.0$

$\dfrac{\sin 86.3°}{107.8} = \dfrac{\sin 56.7°}{|\vec{T_2}|}$

$|\vec{T_2}| = 90.3$

The tensions are 65.0 N and 90.3N in the direction of the wires

3. A plane has a velocity of 450 km/h [$N17°E$]. The wind is 75 $km/$h from [$N30°W$]. Find the ground velocity of the plane.[402.6 $km/$h [$N24.83°E$]]



Let the ground velocity of the plane be $\vec{g}$

$|\vec{g}|^2 = 450^2 + 75^2 - 2(450)(75)\cos 47°$

$|\vec{g}| = 402.604$

$|\vec{g}| = 403$

Using sine law

$\dfrac{\sin\theta}{75} = \dfrac{\sin 47°}{403}$

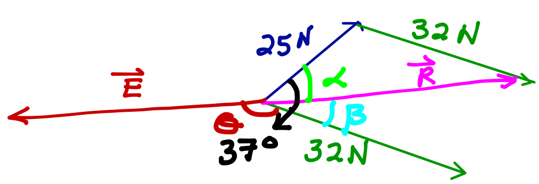
$\sin\theta = \dfrac{75\sin 47°}{403}$

$\theta = 7.83°$

Bearing $= 17° + 7.83°$
$= 24.8°$

∴ The ground velocity is 403 km/h on a bearing of 24.8° or in the direction of N24.8°E

4. A force of 25 N makes an angle of 37° with a force of 32N. Find the magnitude of the equilibrant and the angle it makes with respect to the 32N force. [54.1 N 16.15°]



Let the resultant force be $\vec{R}$

Using cosine law

$|\vec{R}|^2 = 25^2 + 32^2 - 2(25)(32)\cos 143°$

$|\vec{R}| = 54.1$

Using sine law

$\dfrac{\sin\alpha}{32} = \dfrac{\sin 143°}{54.1}$

$\alpha = 20.853°$

$\beta = 37° - 20.85$

$\beta = 16.15°$

$\theta = 180° - \beta$

$\theta = 163.85°$

The equilibrant force is 54 N and makes an angle of 163.85° with the 32 N in the counterclockwise direction

5. If $\vec{a} = [1, 2, 4]$ and $\vec{b} = [2, -3, 1]$ find $\hat{n}$. [$\hat{n} = \frac{1}{\sqrt{230}}[-10, -9, -7]$]

$$\vec{a} \times \vec{b} = [1, 2, -4] \times [2, -3, 1]$$
$$= [2-12, -8-1, -3-4]$$
$$= [-10, -9, -7]$$

$$\left\{\begin{array}{cccccc} 1 & 2 & -4 & 1 & 2 & -4 \\ 2 & -3 & 1 & 2 & -3 & 1 \end{array}\right\}$$

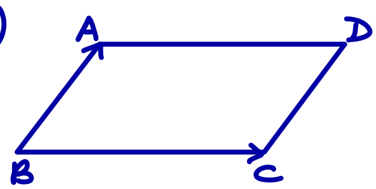
$$|\vec{a} \times \vec{b}| = \sqrt{(-10)^2 + (-9)^2 + (-7)^2}$$
$$= \sqrt{230}$$

Let $\hat{n}$ be a unit vector

$$\hat{n} = \left[\frac{-10}{\sqrt{230}}, \frac{-9}{\sqrt{230}}, \frac{-7}{\sqrt{230}}\right]$$

6. Given $(1, 2, 3)$, $B(3, -1, -2)$, and $C(4, -1, -1)$.
   a. Find D such that ABCD is a parallelogram.
   b. Find the area of ΔABC. [4.09 *units*2]

a)



$$\vec{BA} = \vec{OA} - \vec{OB}$$
$$\vec{BA} = [1, 2, 3] - [3, -1, -2]$$
$$\vec{BA} = [-2, 3, 5]$$

$$\vec{BA} = \vec{CD}$$
$$\vec{CD} = \vec{OD} - \vec{OC}$$
$$[-2, 3, 5] = [x, y, z] - [4, -1, -1]$$
$$[-2, 3, 5] = [x-4, y+1, z+1]$$

$x - 4 = -2$ $\quad$ $y + 1 = 3$ $\quad$ $z + 1 = 5$

$\boxed{x = 2}$ $\quad$ $\boxed{y = 2}$ $\quad$ $\boxed{z = 4}$

$D(2, 2, 4)$

b) $\vec{BA} = [-2, 3, 5]$ $\quad$ $\vec{BC} = [1, 0, 1]$

$$\text{Area of } \Delta ABC = \frac{|\vec{BA} \times \vec{BC}|}{2}$$
$$= \frac{|[-2, 3, 5] \times [1, 0, 1]|}{2}$$
$$= \frac{|[3, 7, -3]|}{2}$$

$$\left\{\begin{array}{cccccc} -2 & 3 & 5 & -2 & 3 & 5 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right\}$$

$$= \frac{\sqrt{3^2 + 7^2 + (-3)^2}}{2}$$
$$= \frac{\sqrt{67}}{2}$$
$$= 4.09$$

Area is 4.09 units^2

7. Given $\vec{a} = [1, -1, -2]$, $\vec{b} = [2, -3, -2]$, and $\vec{c} = [3, -2, 4]$. Find the volume of a parallelepiped whose sides are represented by the given vectors.

$$\text{volume} = |\vec{a} \cdot \vec{b} \times \vec{c}|$$

$$= |[1,-1,-2] \cdot [2,-3,-2] \times [3,-2,-4]|$$

$$= |[1,-1,-2] \cdot [12-4, -6+8, -4+9]|$$

$$= |[1,-1,-2] \cdot [8,2,5]|$$

$$= |8-2-10|$$

$$\begin{array}{cccccc} 2 & -3 & -2 & 2 & -3 & -2 \\ 3 & -2 & -4 & 3 & -2 & -4 \end{array}$$

$$= |-4|$$

$$= 4$$

volume is 4 units^3

1. Alice pulls the handle of a wagon with a force of 200N. If the handle forms a 43° angle with the **vertical**, what is the horizontal component of this force?

2. While camping in northern regions at night, people often keep food out of the reach of animals by hanging it between two trees. If a food bag weighing 435 N is tied between two trees 6 m apart by two ropes that are 4 m and 5 m long (after tying) find the tension in each rope.

3. A pilot in an airplane with an airspeed of 625 km/h wishes to fly a city 1500 km due east. There is a wind blowing from [N 25° E] at a speed of 70 km/h.

   a) In what heading should the pilot steer?

   b) What will be the groundspeed of the airplane?

   c) How long will the trip take?

4. A boat crosses a river and arrives at a point directly across from its starting point. The boat can travel at 3.5 m/s and the current is 1.2 m/s. If the river is 450 m wide at the crossing point how long will it take to cross and in what direction must the boat steer?

5. Suppose 2000 J of work is done by pulling a toboggan 260 m by a force applied at an angle of 40° with the horizontal. What is the magnitude of the pulling force?

6. Consider the points $A(1, 0, 2)$, $B(2, 0, 1)$, $C(3, 2, -1)$. If a force of 10N acts in the direction of $[1, 1, -1]$ to move an object from A to B, and distance is measured in meters, how much work is done?

7. An airplane pilot checks her instruments and finds that the speed of the plane relative to the air is 325 km/h. The instruments also show that the plane is pointed in a direction [N30°W]. A radio report indicates that the wind velocity is 80 km/h blowing from [E 25° N]. What is the velocity of the plane relative to the ground as it is recorded by an air traffic controller in a nearby airport?

8. A large cruise boat is moving at 15 km/h [E25°S] relative to the water. A person jogging on the ship moves across the ship in a northerly direction at 6 km/h. What is the velocity of the jogger relative to the water?

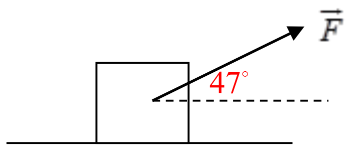
9. A plane is seen to travel in a direction [N55°E]. If its ground velocity was 300 km/h and the wind was blowing 50 km/h from [N45°W], what was the plane's velocity relative to the air?

10. An object weighing 20 kg is suspended by two wires of equal length 50 cm. How far apart must they be attached to the surface above so that the force on each is 150 N?

11. Two vectors, $\vec{a}$ and $\vec{b}$,of magnitude 3 and 5, respectively, make an angle of 57° with each other. Determine the magnitude and direction of $\vec{b}-\vec{a}$.(Round your answer to one decimal place) [3 marks]

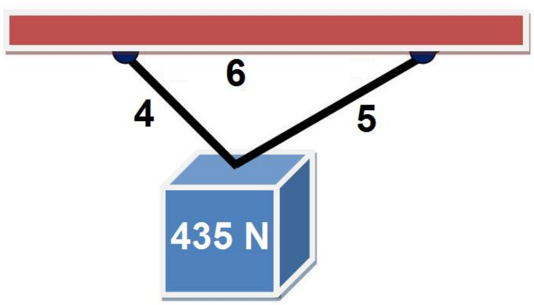
MCV4UE Relax…Review

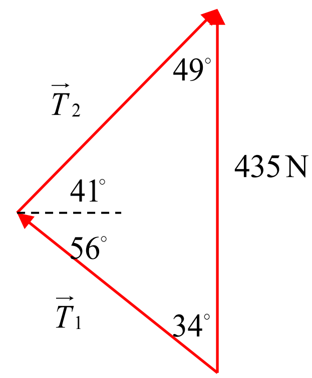
1. Alice pulls the handle of a wagon with a force of 200N. If the handle forms a 43° angle with the **vertical**, what is the horizontal component of this force?



$$\left|\vec{F}_y\right| = \left|\vec{F}\right|\cos 47^{\circ}$$
$$= 200\cos 47^{\circ}$$
$$= 136.4\text{ N}$$

2. While camping in northern regions at night, people often keep food out of the reach of animals by hanging it between two trees. If a food bag weighing 435 N is tied between two trees 6 m apart by two ropes that are 4 m and 5 m long (after tying) find the tension in each rope.





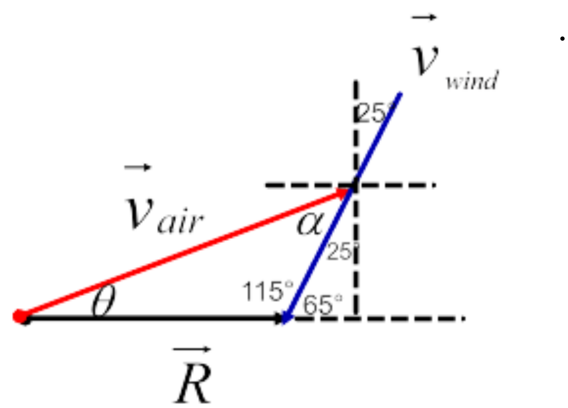
$$\cos\theta = \frac{6^2+4^2-5^2}{2(6)(4)}$$
$$\theta = 56^{\circ}$$
$$\cos\alpha = \frac{6^2+5^2-4^2}{2(6)(5)}$$
$$\alpha = 41^{\circ}$$
$$\frac{\sin 97^{\circ}}{435} = \frac{\sin 34^{\circ}}{\left|\vec{T}_2\right|} = \frac{\sin 49^{\circ}}{\left|\vec{T}_1\right|}$$
$$\left|\vec{T}_1\right| = 330.8\text{ N},\ \left|\vec{T}_2\right| = 245.1\text{N}$$

3. A pilot in an airplane with an airspeed of 625 km/h wishes to fly a city 1500 km due east. There is a wind blowing from [N 25º E] at a speed of 70 km/h.

a) In what heading should the pilot steer?

b) What will be the groundspeed of the airplane?

c) How long will the trip take?



$|\vec{v}_{air}| = 625 \text{ km/h}$

$d = 1500\text{km}$

$|\vec{v}_{wind}| = 70 \text{ km/h}$

a) $\dfrac{\sin 115^\circ}{625} = \dfrac{\sin\theta}{70}$

$\theta = 5.8^\circ$

b) $\dfrac{\sin 59.2^\circ}{|\vec{R}|} = \dfrac{\sin 115^\circ}{625}$

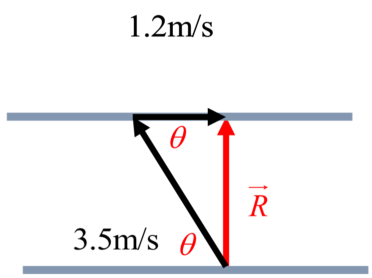
$|\vec{R}| = 592.3 \text{ km/h}$

c) $t = \dfrac{d}{v}$

$t = \dfrac{1500}{592.3}$

$t = 2.53 \text{ hr}$

4. A boat crosses a river and arrives at a point directly across from its starting point. The boat can travel at 3.5 m/s and the current is 1.2 m/s. If the river is 450 m wide at the crossing point how long will it take to cross and in what direction must the boat steer?



$$\theta = \cos^{-1}\left(\frac{1.2}{3.5}\right)$$
$$\theta = 70^\circ$$
$$|\vec{R}| = \sqrt{3.5^2 - 1.2^2}$$
$$|\vec{R}| = 3.3\,m/s$$
$$t = \frac{d}{v}$$
$$t = \frac{450}{3.3} = 136s$$

5. Suppose 2000 J of work is done by pulling a toboggan 260 m by a force applied at an angle of 40° with the horizontal. What is the magnitude of the pulling force?

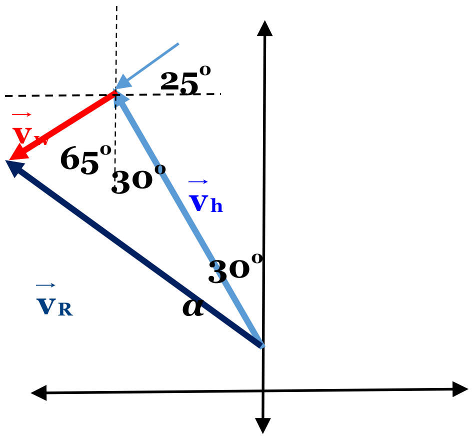
$$W = |\vec{F}||\vec{d}|\cos(\theta)$$
$$2000 = |\vec{F}|(260)\cos(40^\circ)$$
$$|\vec{F}| = 10.04N$$

6. Consider the points $A(1, 0, 2)$, $B(2, 0, 1)$, $C(3, 2, -1)$. If a force of 10N acts in the direction of $[1,1,-1]$ to move an object from A to B, and distance is measured in meters, how much work is done?

$$\vec{F} = k[1,1,-1]$$
$$|\vec{F}| = k\sqrt{1+1^2+(-1)^2}$$
$$10 = k\left(\sqrt{3}\right) \rightarrow k = \frac{10\sqrt{3}}{3}$$
$$\vec{F} = \frac{10\sqrt{3}}{3}[1,1,-1]$$

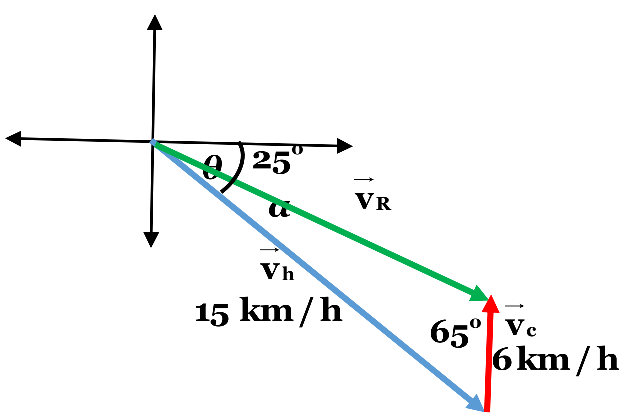
$$w = \vec{F}\bullet\vec{d}$$
$$= \frac{10\sqrt{3}}{3}[1,1,-1]\bullet[1,0,-1]$$
$$= \frac{20\sqrt{3}}{3}J$$

7. An airplane pilot checks her instruments and finds that the speed of the plane relative to the air is 325 km/h. The instruments also show that the plane is pointed in a direction [N30°W]. A radio report indicates that the wind velocity is 80 km/h blowing from [E 25° N]. What is the velocity of the plane relative to the ground as it is recorded by an air traffic controller in a nearby airport?



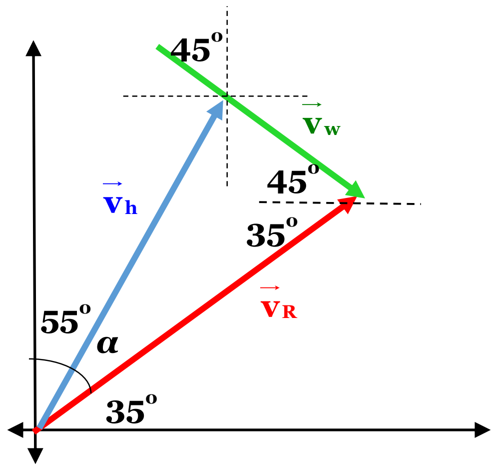
$$\left|\vec{v}_w\right| = \sqrt{325^2 + 80^2 - 2(325)(80)\cos(95^\circ)}$$
$$\doteq 341.1\ \text{km/h}$$
$$\frac{\sin(\alpha)}{80} = \frac{\sin(95^\circ)}{341.4}$$
$$\sin(\alpha) = \frac{80\sin(95^\circ)}{341.4}$$
$$\alpha \doteq 13.5^\circ$$
$$\therefore \vec{v}_R = 341.4\ \text{km/h [N43.5°W]}$$

8. A large cruise boat is moving at 15 km/h [E25°S] relative to the water. A person jogging on the ship moves across the ship in a northerly direction at 6 km/h. What is the velocity of the jogger relative to the water?



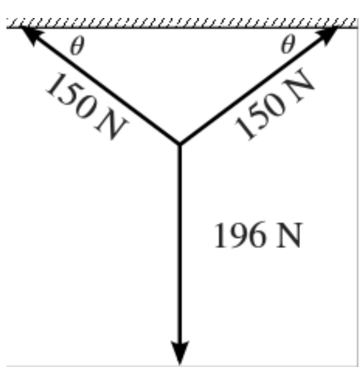
$$\left|\vec{v}_c\right| = \sqrt{6^2 + 15^2 - 2(6)(15)\cos(65^\circ)}$$
$$\doteq 13.6\ \text{km/h}$$
$$\frac{\sin(\alpha)}{6} = \frac{\sin(65^\circ)}{13.6}$$
$$\sin(\alpha) = \frac{6\sin(65^\circ)}{13.6}$$
$$\alpha \doteq 23.6^\circ$$
$$\theta = 25^\circ - 23.6^\circ$$
$$\doteq 1.4^\circ$$
$$\therefore \vec{v}_R = 13.6\ \text{km/h [E1.4°S]}$$

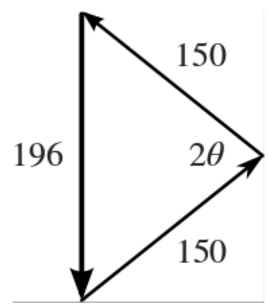
9. A plane is seen to travel in a direction [N55°E]. If its ground velocity was 300 km/h and the wind was blowing 50 km/h from [N45°W], what was the plane's velocity relative to the air?

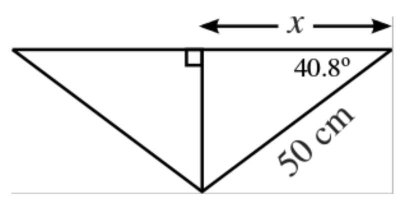


$$\left|\vec{v}_h\right| = \sqrt{300^2 + 50^2 - 2(300)(50)\cos(80^\circ)}$$
$$\doteq 295.4 \text{ km/h}$$
$$\frac{\sin(\alpha)}{50} = \frac{\sin(80^\circ)}{295.4}$$
$$\sin(\alpha) = \frac{50\sin(80^\circ)}{295.4}$$
$$\alpha \doteq 9.6^\circ$$
$$\theta = 55^\circ - 9.6^\circ$$
$$\doteq 45.4^\circ$$
$$\therefore \vec{v}_h = 295.4 \text{ km/h [N}45.4^\circ\text{E]}$$

10. An object weighing 20 kg is suspended by two wires of equal length 50 cm. How far apart must they be attached to the surface above so that the force on each is 150 N?



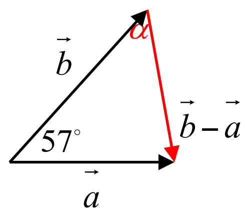




$$\cos(2\theta) = \frac{150^2 + 150^2 - 196^2}{2 \times 150 \times 150}$$
$$2\theta = 81.6^\circ$$
$$\theta = 40.8^\circ$$

$$\cos(40.8^\circ) = \frac{x}{50}$$
$$x = 37.85$$
$$2x = 75.7$$

11. Two vectors, $\vec{a}$ and $\vec{b}$,of magnitude 3 and 5, respectively, make an angle of 57° with each other. Determine the magnitude and direction of $\vec{b}$-$\vec{a}$.(Round your answer to one decimal place) [3 marks]



$$\left|\vec{b}-\vec{a}\right| = \sqrt{\left|\vec{b}\right|^2 + \left|\vec{a}\right|^2 - 2\left(\left|\vec{b}\right|\left|\vec{a}\right|\right)\cos 57^\circ}$$

$$\left|\vec{b}-\vec{a}\right| = 4.2 \text{ units}$$

$$\frac{\sin\alpha}{3} = \frac{\sin 57^\circ}{4.2} \rightarrow \angle\alpha = 36.8^\circ$$

$$\therefore \vec{b}-\vec{a} = 4.2 \text{ units } 36.8^\circ \text{ to } \vec{a} \text{ away from } \vec{b}$$