

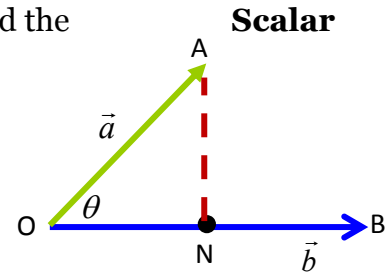


Unit 4
Algebraic Vectors and
Their Applications

Applications of the Dot Product

Part I. Scalar and Vector Projections

Given two vectors, \vec{a} and \vec{b} , placed tail to tail with angle θ between them, drop a perpendicular from the tip of \vec{a} to the line containing \vec{b} . The vector lying along the line containing \vec{b} , which has magnitude equal to the component of \vec{a} in the direction of \vec{b} (i.e., \overline{ON} in our diagram), is called the **vector projection** of \vec{a} onto \vec{b} . The **magnitude of the vector projection** of \vec{a} onto \vec{b} is called the **Scalar Projection**.



A. Scalar Projections – no direction

| $0^\circ \leq \theta \leq 90^\circ$ (same direction as \vec{b}) | $90^\circ < \theta \leq 180^\circ$ (opposite direction as \vec{b}) |
|--|---|
| | |

- i. Scalar Projection of \vec{a} onto \vec{b}

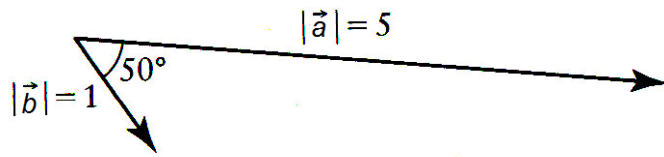
$$\left| \text{Proj}_{\vec{b}} \vec{a} \right| = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

- ii. Scalar Projection of \vec{b} onto \vec{a}

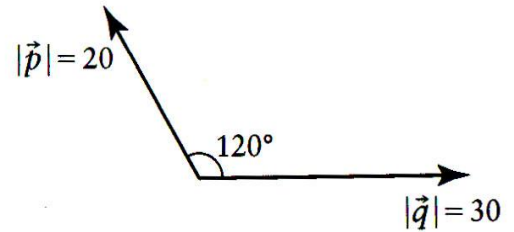
$$\left| \text{Proj}_{\vec{a}} \vec{b} \right| =$$

Example 1:

a) Determine the scalar projection of \vec{a} onto \vec{b} .



b) Determine the scalar projection of \vec{p} onto \vec{q} .



B. Vector Projections – have direction

The *vector* projection of \vec{a} onto \vec{b} is $Proj_{\vec{b}} \vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \right) \vec{b}$

The *vector* projection of \vec{b} onto \vec{a} is $Proj_{\vec{a}} \vec{b} =$

Example 2: Given vectors $\vec{a} = (-4, 1)$ and $\vec{b} = (4, 3)$, determine $Proj_{\vec{a}} \vec{b}$.

Example 3: Let $\vec{u} = proj_{\vec{b}} \vec{a}$ where $\vec{a} = (1, 1)$ and $\vec{b} = (4, 2)$. Show that $|\vec{a} - \vec{u}| \leq |\vec{a} - k\vec{b}|$ for all $k \in \mathbb{R}$.

Example 4: The scalar projection of vector $\vec{u} = [1, m, 0]$ onto vector $\vec{v} = [2, 2, 1]$ is 4. Determine the value of m .

Example 5: The vector \vec{r} is twice as long as the vector \vec{s} . The angle between the vectors is 120° . The vector projection of \vec{s} on \vec{r} is $[2, -1, 7]$. Determine \vec{r} .

Part II. Work

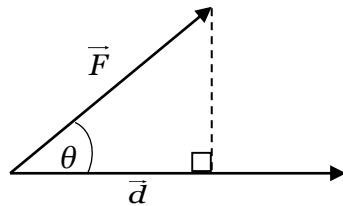
Definition: In Physics, WORK is done whenever a force, applied to an object, causes a displacement in the object from one position to another.

- WORK is equal to the displacement traveled multiplied by the magnitude of the applied force in the direction of motion.

For instance, if the force is in the same direction as the displacement, then just multiply the magnitudes.



However if the force acts at an angle to the $work = |\vec{F}||\vec{d}|$ displacement vector, we use the component of the force, in the direction of the displacement vector (i.e we use the projection of \vec{F} onto \vec{d})



| |
|----------|
| $work =$ |
|----------|

The work done on an object is the dot product of the force applied on the object, and the displacement of the object.

Note: - Work is a scalar quantity. The unit of measurement is the **Joule (J)** or Newton-metre (N-m).

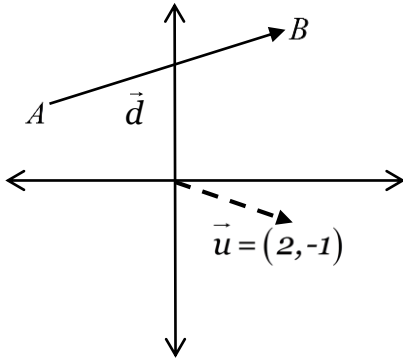
- No matter how much force is applied, if no displacement occurs, work = 0.

Example 1: A crate, on a ramp is hauled 8m up the ramp by a constant force of 20N applied at an angle of 30° to the ramp. Calculate the work done by the force.

Example2: A shopper in a supermarket pushes a cart with a force of 35 N directed at an angle of 25° downward from the horizontal. Find the work done by the shopper as she moves down an aisle 50 m in length.

Example 3: A crate with a weight of 57 N rests on a frictionless ramp inclined at an angle of 30° to the horizontal. What force must be applied at an angle of 20° to the ramp so that the crate remains at rest?

Example 4: A force 15N acting along the vector $\vec{u} = (2, -1)$, displaces a particle from A(-4, 2) to B(1, 5). If the distance is in meters, calculate the amount of work done.

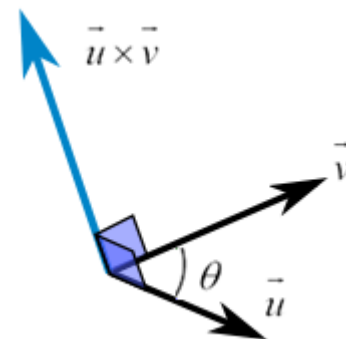


Practice

1. An object is dragged 5m up a ramp under a constant force of 30N applied at an angle 30° to the ramp. Find the work done.
2. A man in a wheelchair moves 15m down a ramp inclined at an angle of 10° to the horizontal. The mass of the man and the wheelchair together is 80kg. ($1\text{kg} = 9.8\text{N}$) Calculate the work done.
3. An object is dragged 5m on level ground by a 20N force that is applied 50° to the ground. It is then dragged 8m up a ramp with the same force. The inclination of the ramp is 30° to the ground. At the top of the ramp, the object is dragged, with the same force, horizontally 13m. Find the total work done.
4. A box is lifted through a distance of 1.2 m and placed on a wagon by exerting a force of 105 N. The wagon is then pulled through a distance of 25 m by a 45 N force applied at an angle of 35° to the ground. Find the total work done.
5. Determine the work done by a force of magnitude 55N acting in the direction of the vector $\vec{u} = (2, -2, 1)$, which moves an object from A(1,4,-1) to B (-1 ,2,1).The distance is in metres.

Cross Product of 2 Vectors $\vec{u} \times \vec{v}$ – in R^3

- not multiply, slightly bigger
- also known as vector product
- result is always a vector not a scalar
- cross product is a particular vector that's perpendicular to 2 non-collinear vectors, in fact, **there's an infinite number of such vectors!**



Cross Product – Algebraic Vectors

Given the vectors $\vec{u} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ and $\vec{v} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ then:

1) Set up the vector components in the following manner:

$$\text{For } \vec{u} \times \vec{v}: \quad \begin{array}{cccc} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_1 \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1 \end{array}$$

2) To determine the x , y , and z component:

x -component of $\vec{u} \times \vec{v}$, conduct the following operation on the **middle** four terms:

$$\begin{vmatrix} \mathbf{u}_2 & \mathbf{u}_3 \\ \mathbf{v}_2 & \mathbf{v}_3 \end{vmatrix} = \mathbf{u}_2 \mathbf{v}_3 - \mathbf{u}_3 \mathbf{v}_2$$

y -component: conduct the same operation, but on the four terms on the **right**:

$$\begin{vmatrix} \mathbf{u}_3 & \mathbf{u}_1 \\ \mathbf{v}_3 & \mathbf{v}_1 \end{vmatrix} = \mathbf{u}_3 \mathbf{v}_1 - \mathbf{u}_1 \mathbf{v}_3$$

z -component: repeat for the four terms on the **left**:

$$\begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 \\ \mathbf{v}_1 & \mathbf{v}_2 \end{vmatrix} = \mathbf{u}_1 \mathbf{v}_2 - \mathbf{u}_2 \mathbf{v}_1$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{vmatrix}$$

$$\vec{u} \times \vec{v} =$$

Example 1: Find the cross product of $\vec{u} = [-2, 1, -4]$ and $\vec{v} = [3, 0, -1]$.

Example 2: Find the cross product of $\vec{u} = [-2, 1, -4]$ and $\vec{v} = [6, -3, 12]$.

Example 3: If $\vec{a} = [1, 3, -1]$, $\vec{b} = [2, 1, 5]$ and $\vec{v} = [-3, y, z]$ $\vec{a} \times \vec{v} = \vec{b}$, find ***y*** and ***z***.

Properties of the Cross Product

The Cross Product is:

1) **Anti-Commutative:** $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

2) **Distributive** over vector addition: $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

3) **Associative** over scalar multiplication: $m(\vec{u} \times \vec{v}) = (m\vec{u}) \times \vec{v} = \vec{u} \times (m\vec{v}), m \in \mathbb{R}$

4) If \vec{u} and \vec{v} are non-zero, $\vec{u} \times \vec{v} = \vec{0}$ if and only if \vec{u} and \vec{v} are collinear.

Example 4: If ABC is a triangle with vertices A(1, 1, -1), B(1, 0, 1), and C(1 + k, 0, 2) and $\vec{AB} \times \vec{AC} = [-1, 2, 1]$, find the value of k.

Example 5: Determine the value of m and n for $\vec{a} = [m, -12, 9]$ and $\vec{b} = [5, n, -3]$ such that

$\vec{a} \times \vec{b} = \vec{0}$. What is the relationship between \vec{a} and \vec{b} .

Magnitude of the Cross Product

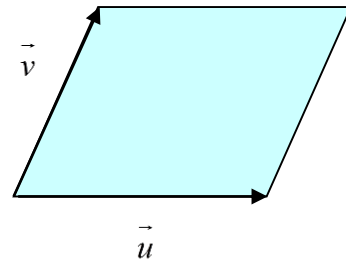
The magnitude of the cross product is defined according to the following equation:

$$\boxed{|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta}$$

, where θ is the angle between the vectors such that $0^\circ \leq \theta \leq 180^\circ$.

It represents the area of the parallelogram enclosed by the two vectors.

$$A_{\text{parallelogram}} = |\vec{u} \times \vec{v}|$$



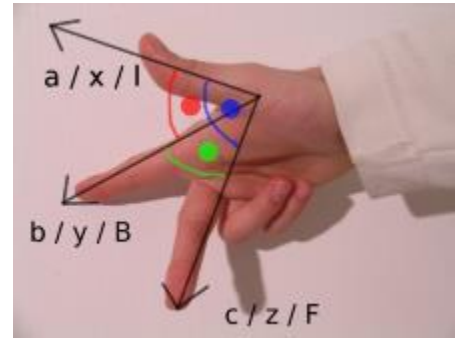
Example 6: Three vertices of a **parallelogram ABCD** are **A(3,-1,2)**, **B(1,2,-4)** and **C(-1,1,2)**.

- a) find the coordinate of the fourth vertex.
- a) find the area of triangle ABC.

Direction of the Cross Product – Into the Page or Out of the Page

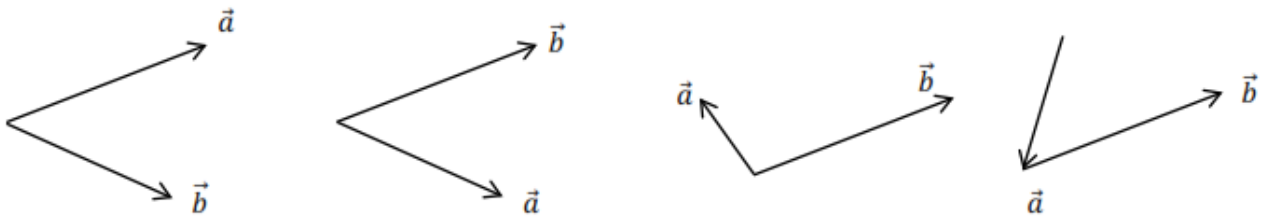
Recall that the cross product gives us a vector that is perpendicular to two vectors. To determine whether the cross product $\vec{a} \times \vec{b}$ is into the page or out of the page, we use the Right Hand Rule.

- Make your thumb lie on the first vector (\vec{a}).
- Make your index finger lie on the second vector (\vec{b})
- Make your middle finger perpendicular to your thumb



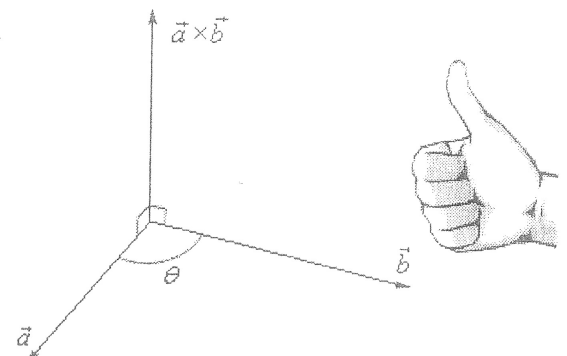
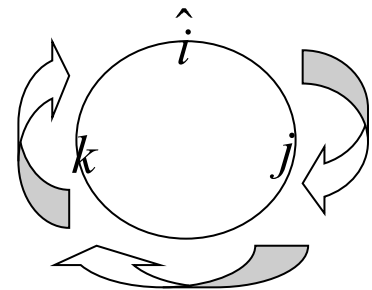
Example 7: Given the following vectors determine if the cross product is into the page or out of the page.

- NOTE: Just like the dot product, **vectors must be tail to tail** when evaluating a cross product.



Example 8: Determine the following cross products using the correct sign convention

- | | |
|--|--|
| a) $\hat{i} \times j = \underline{\hspace{2cm}}$ | d) $k \times j = \underline{\hspace{2cm}}$ |
| b) $k \times \hat{i} = \underline{\hspace{2cm}}$ | e) $j \times \hat{i} = \underline{\hspace{2cm}}$ |
| c) $\hat{i} \times k = \underline{\hspace{2cm}}$ | f) $j \times k = \underline{\hspace{2cm}}$ |



Curl the fingers of your right hand from the first vector to the second. The thumb then points in the direction of the cross product of the two vectors.

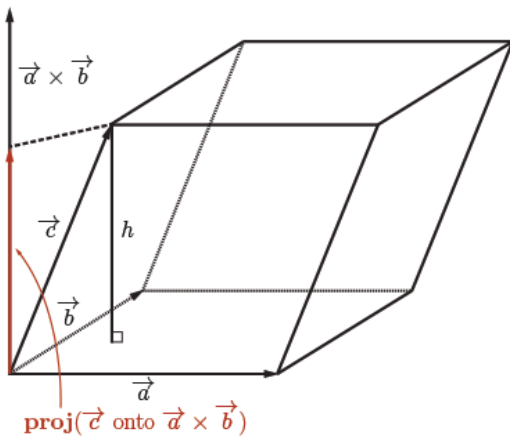
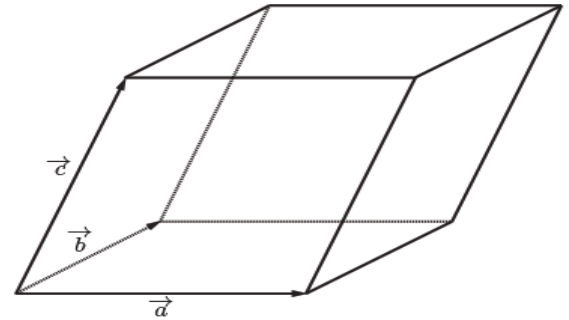
Applications of the Dot Product and Cross Product

I. Volume of Parallelepiped

A **parallelepiped** is a box-like solid, where the opposite faces of which are parallel and congruent parallelograms.

Let \vec{a} , \vec{b} , and \vec{c} be three vectors whose tails meet at one vertex of the parallelepiped.

The absolute value of the triple scalar product of these three vectors gives the volume of the parallelepiped.



$$\begin{aligned} \text{Volume} &= (\text{area of base}) \times \text{height} \\ &= (\text{area of parallelogram}) \times \text{height} \end{aligned}$$

parallelogram is made up of vector \vec{a} and \vec{b} so its area = $|\vec{a} \times \vec{b}|$

The height = the magnitude of the projection of \vec{c} onto the vector perpendicular to the base: $|\text{Proj}_{\vec{a} \times \vec{b}} \vec{c}| = \frac{\vec{c} \cdot (\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$

$$\text{Volume} = (\text{area of parallelogram}) \times \text{height}$$

$$V = |\vec{c} \cdot (\vec{a} \times \vec{b})|$$

Question: Is $|\vec{c} \cdot (\vec{a} \times \vec{b})|$ equivalent to $|\vec{a} \cdot (\vec{b} \times \vec{c})|$? _____.

Triple Scalar Product: is called the quantity $\vec{c} \cdot (\vec{a} \times \vec{b})$, since it returns a scalar value.

Definition: Three vectors \vec{a} , \vec{b} , and \vec{c} are coplanar if and only if $\vec{c} \cdot (\vec{a} \times \vec{b}) = 0$

Example 9: Determine the volume of a parallelepiped given the vectors $\vec{a} = [-2, 2, 5]$, $\vec{b} = [0, 4, 1]$ and $\vec{c} = [0, 5, -1]$.

Example 10: Determine if the vectors $[1, 3, 2]$, $[5, 0, -1]$, and $[-4, 3, 3]$ are coplanar.

Example 11. Circle whether the following expressions are vectors, scalar, or meaningless.

a) $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{c})$ vector scalar meaningless

b) $\frac{(\vec{a} \cdot \vec{b}) \vec{c} \times (\vec{a} \times \vec{b})}{|\vec{c}|}$ vector scalar meaningless

c) $\vec{a} \times \vec{b} + \vec{u} \cdot \vec{c}$ vector scalar meaningless

d) $\frac{\vec{a} \cdot \vec{b} + \vec{a} \cdot (\vec{a} \times \vec{b})}{|\vec{b}|}$ vector scalar meaningless

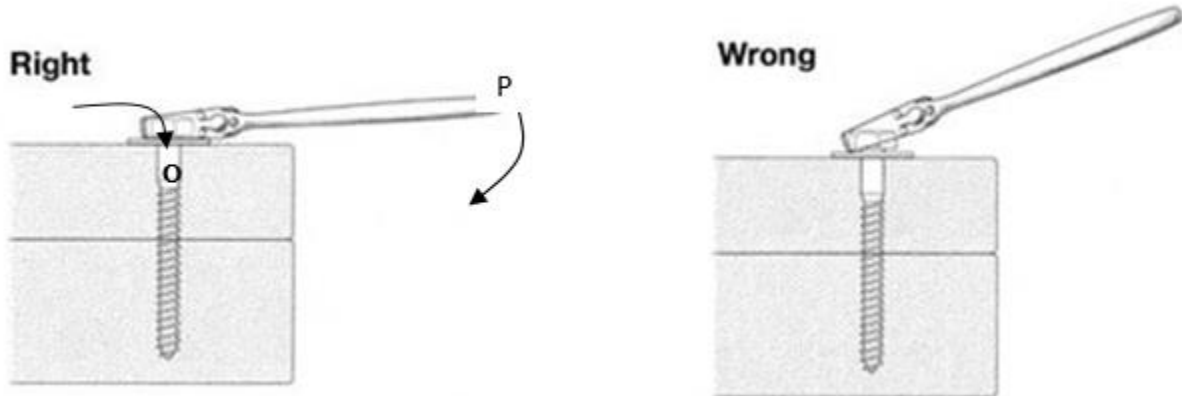
II. Torque

Torque

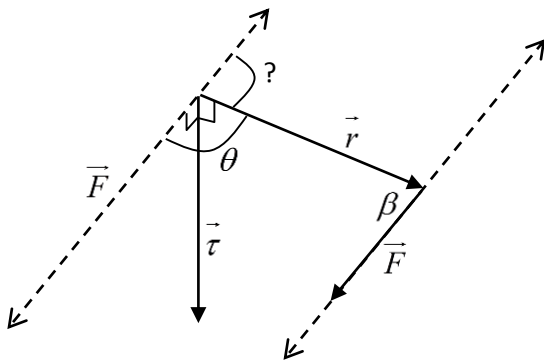
Definition: Torque can be considered the turning effect of a force on an object.

It has a magnitude, measured in Newton-metres (Nm) and a direction. It is therefore, a vector value.

For example: Turning a bolt with a wrench to drive it into a block of wood.



To find θ , we arrange \vec{r} and \vec{F} tail to tail.



Note: Torque is max when $\theta = 90^\circ$
The most efficient way to maximize torque with a certain force is to maximize \vec{r} AND/OR $\sin(\theta)$

The torque vector will always act in a direction **perpendicular** to both \vec{r} and \vec{F} .

In our example the bolt is either being pushed into the block of wood (moving into the board) or being pulled out of the block of wood (moving out of the board). In both cases, the motion is orthogonal to the applied force and to the lever arm.

(I) Therefore, the torque produced can be determined by finding the cross product of \vec{r} and \vec{F} .

$$\vec{\tau} = \vec{r} \times \vec{F}$$

(II) It follows then that the magnitude of the torque produced is the magnitude of the cross product of \vec{r} and \vec{F} .

$$|\vec{\tau}| = |\vec{r} \times \vec{F}|$$

or

$$|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin \theta$$

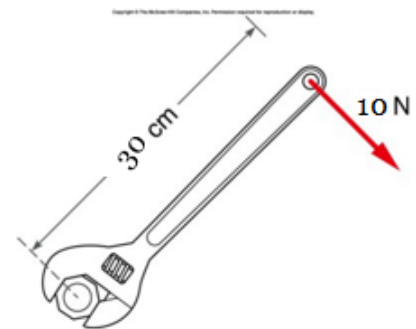
***Recall:** θ is the angle between the \vec{r} and \vec{F} when arranged tail to tail.

Its magnitude measures the twisting effect of the force, while its direction gives the direction of the axis through O about which the force tends to twist (i.e. down and clockwise with a right-hand thread or use the right-hand rule).

The direction of the torque vector is found by using the right hand rule.

***Terminology:** i. Orthogonal ii. Normal iii. Fulcrum iv. “tighten bolt” vs. “loosen bolt” (into/out of paper)

Example1: A 10 N force is applied at the end of a 30cm wrench with which it makes a 60° angle. Calculate the magnitude of the torque.



Example 2: A 50N force is applied at a point on a door that is 70cm from the side of the hinged edge. The force makes a 30° angle with the door. Calculate the magnitude and describe the direction of the torque vector. Include a diagram.

Example3: A force of 50N is applied to a bike pedal making a 65° angle with the lever arm. If the lever arm is 30cm long, calculate the magnitude of the torque produced.

Example4: A bolt is being rotated by a 20cm wrench. If the wrench is oriented 30° to the horizontal, and a downward force of 30N is being applied to the end of the wrench at an angle 40° to the vertical, find the magnitude of the torque produced. Is the bolt being screwed in or removed?

Applications of Vector Addition – Force

Force : A physical influence that causes a change in direction on a physical object. It is measured in a unit called Newtons (N).

To describe a force it is necessary to state:

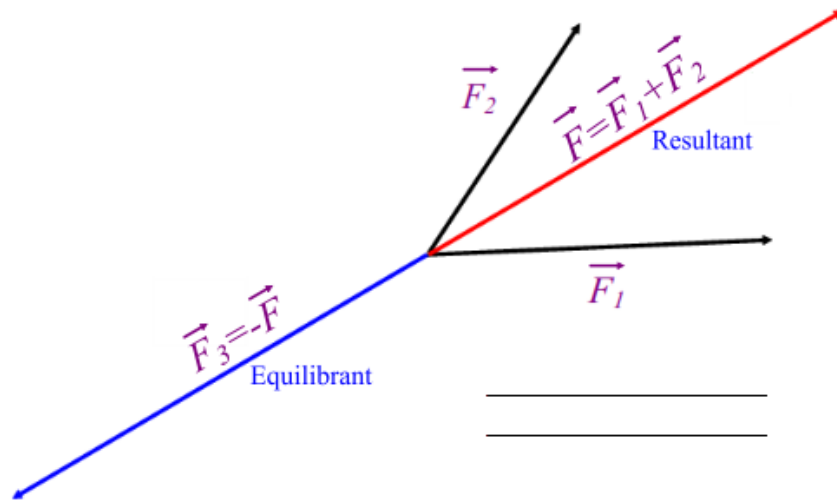
- its direction
- the point at which it is applied
- its magnitude

The *resultant* is the sum of the vectors representing two or more forces.

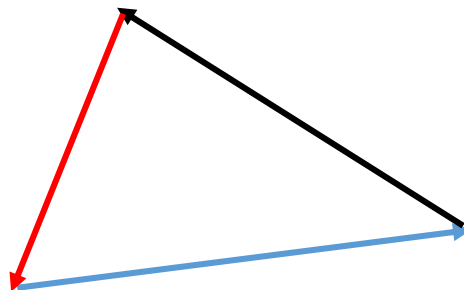
The *equilibrant* is the opposite force that would exactly counterbalance the resultant.

Equilibrant Force: Let \vec{F}_1 and \vec{F}_2

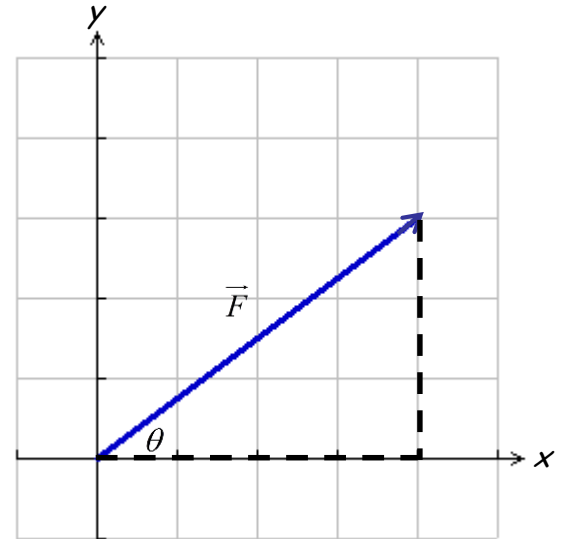
be two forces acting upon an object. The resultant vector can be represented by a third vector using the concepts from vector addition.



For a system of three forces to be in equilibrium the vectors representing those forces, when placed _____.

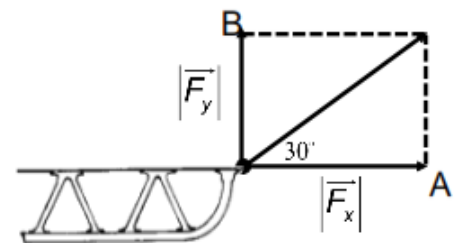


RESOLVING VECTORS INTO COMPONENTS (used in application problems when a Cartesian coordinate system is not used)



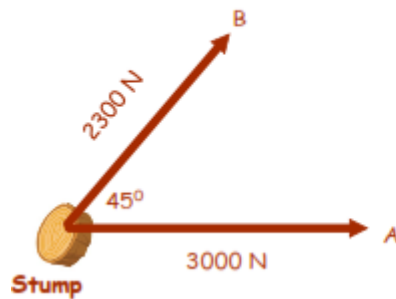
Ex. 1 A sleigh is being pulled with a 5N force at an angle of 30° with the ground.

- Calculate the force that is pulling the sleigh forward.
- the force that tends to lift the sleigh.



Ex. 2: Two tractors are being used to pull a tree stump out of the ground. The larger tractor pulls with a force of 3000 N[E]. The smaller tractor pulls with a force of 2300 N [NE]. Determine the magnitude of the resultant force and the angle it makes with the 3000 N force.

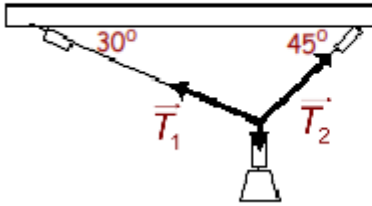
position diagram



Vector diagram

Ex. 3: A 100 N weight is suspended from the ceiling by two ropes that make angles of 30° and 45° with the ceiling. Determine the tension in each rope.

position diagram



Vector diagram

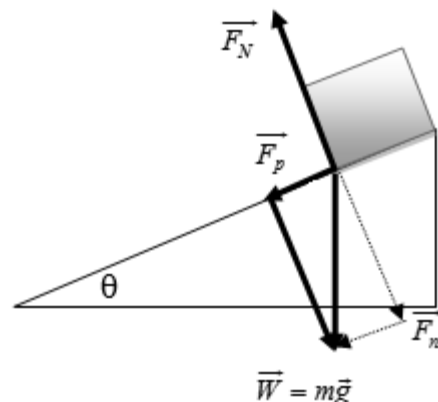
Ex. 4: Three forces having magnitudes of 4 N, 5 N, and 7 N are in a state of equilibrium. Calculate the angle between the two smaller forces.

Ex. 5: A 10 kg mass is supported by two strings of length 5 m and 7 m attached to two points in the ceiling 10 m apart. Find the tension in each string.

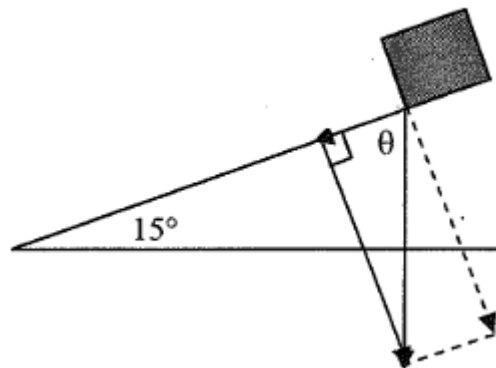
A Ramp Problem

The next example shows that rectangular components do not necessarily have to be horizontal or vertical.

A box weighing \vec{W} Newton is resting on a ramp that is inclined at an angle of θ° . Resolve the weight into the rectangular components, \vec{F}_p , the force parallel to the surface, and \vec{F}_n , the force perpendicular to the surface. Note \vec{F}_N is the force of the ramp pushing against the box. This force counteracts the component of gravity in the opposite direction to keep the box at rest.



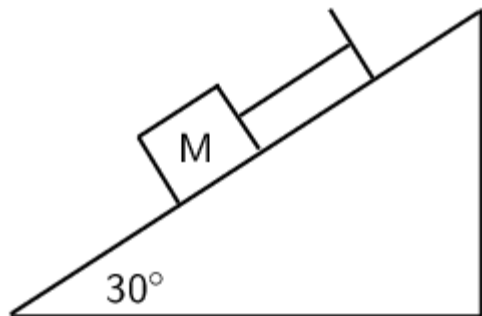
Ex. 7: Components of the forces of gravity A 20-kg trunk is resting on a ramp inclined at an angle of 15° . Calculate the components of the force of gravity on the trunk that are parallel and perpendicular to the ramp.



Ex. 8 : A block of mass M is held stationary by a rope of negligible mass. The block rests on a frictionless plane which is inclined at 30° to the horizontal.

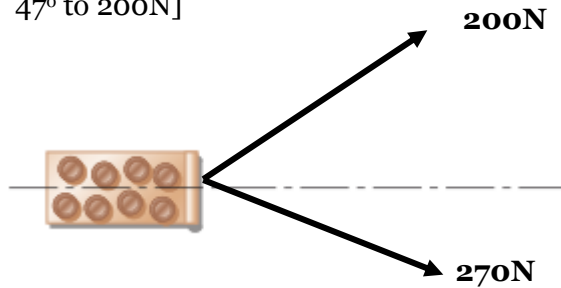
- Resolve the force due to gravity into components that are parallel and perpendicular to the plane.

- Calculate the tension in the rope. ($\vec{g} = 9.8 \text{ m/s}^2$)

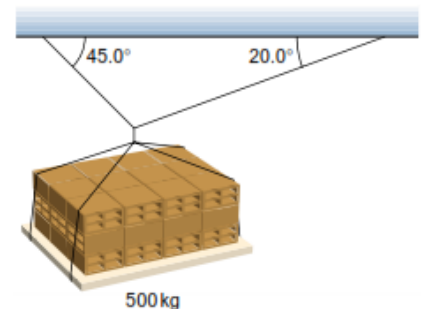


Practice Questions

1. Two horses pull a load. The ropes between the horses and the load are at an angle of 80° to each other. One horse pulls with a force of 200 N (newton), and the other with a force of 270 N. Here is a diagram to illustrate the two forces. Calculate the resultant force. [Ans. 363 N at 47° to 200N]

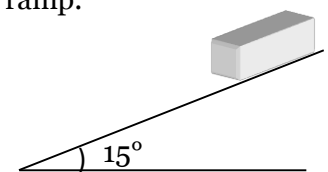


2. A mass of 500 kg is supported by two cables as illustrated. What is the tension in each cable? ($\bar{g} = 9.8 \text{ m/s}^2$)
[Ans. 3823 N and 5080.5N]

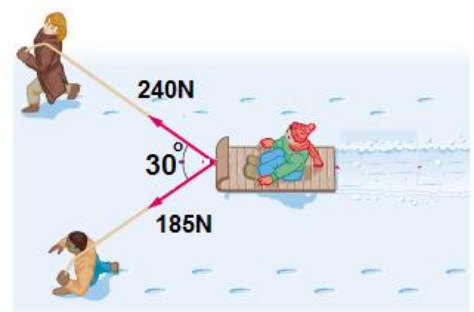


3. A lawnmower is pushed across a lawn by applying a force of 95 N along the handle of the mower. The handle makes an angle of 60.0° with the horizontal.
- What are the horizontal and vertical components of the force? [Ans. $F_x = 48 \text{ N}$, $F_y = 82 \text{ N}$]
 - The handle is lowered so that it makes an angle of 30.0° with the horizontal. What are the horizontal and vertical components of the force? [Ans. $F_x = 82 \text{ N}$, $F_y = 48 \text{ N}$]

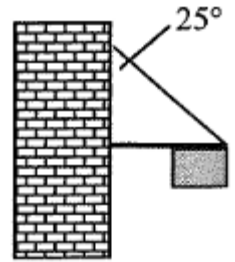
4. 20-kg trunk is resting on a ramp inclined at an angle of 15° . Calculate the components of force of gravity on the trunk that are parallel and perpendicular to the ramp.
[Ans. $\bar{F}_p = 50.7 \text{ N}$, $\bar{F}_n = 189.3 \text{ N}$]



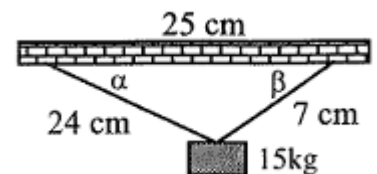
5. Using two ropes that make an angle of 30° to each other, Jack and Alex pull Bill in a sleigh. Jack pulls with 240N force and Alex pulls with a force of 185N. Determine the magnitude and direction of the equilibrant force. [Ans. 410.8 N, makes an angle of 167° with the larger force]



6. An advertising sign is supported by a horizontal steel brace extending at right angles from the side of a building, and by a wire attached to the building above the brace at an angle of 25° . If the force of gravity on the sign is 850 N , find the tension in the wire and the compression in the steel brace.



7. An object of 15 kg is suspended by two cords of lengths 7 cm and 24 cm , and these two cords are 25 cm apart. Find the tension in each cord. [Ans. The tensions of two cords are 141N and 41.3N]



Warm Up

A ski chairlift is suspended between two towers that are 60 m apart horizontally. When the chairlift is 20 m from one tower, the cable sags 1 m . The chairlift is loaded with four skiers with a combined weight of 250 N (including the mass of the chair). What are the tensions on the two parts of the cable?

Velocity as a Vector

Velocity measures the direction and the rate of change in the position of an object.

- Velocity is a vector because it has both _____ and _____.
- **Air speed (water speed)** is the speed of a plane (boat) relative to a person on board.
- **Ground speed** is the speed of a plane (boat) relative to a person **on the ground** and includes the effect of wind (current).

Ex1: A boat with a forward velocity of 14 m/s is traveling across a river, directly towards the opposite shore. At the same time, a current of 5 m/s carries the boat down the river.

(a) Determine the resultant velocity of the boat.

(b) Suppose the river was 100 m across, how long would it take for the boat to cross the river?

Ex2: Alex wishes to use a canoe to cross to the other side of a river, which is 30 m wide. The river is flowing at 10 m/min and Alex can paddle at 20 m/min.

(a) If he points his canoe directly across the river (perpendicular to the bank), where will he land? How long will the crossing take?

(b) In what direction should he aim the canoe in order to land at a point directly opposite his starting point? How long will it take to make this crossing?

Ex3 : (a) Fiona heads straight out across a stream flowing at 2 km/hr. She can row at 3.5 km/hr in still water. Determine her resultant velocity.

(b) Suppose Fiona needs to land on the bank directly opposite her starting position. Which direction would she have to steer and what would be her resultant velocity?

Ex.4 : A small aircraft is flying on a heading $[N\ 30^\circ\ W]$ at a constant speed of 150 km/h. The wind is blowing **from** 5° south of west with a speed of 40 km/h. Determine the actual speed and direction of the aircraft relative to the ground

Ex5 : A pilot wishes to fly from Toronto to Montreal a distance of 500 km on a heading of $[N 75^\circ E]$. The airspeed of the plane is 550 km/h. An 80 km/h wind is blowing from $[N 55^\circ W]$.

- a) What heading should the pilot take to reach his destination?
- b) What will be the speed of the plane relative to the ground?(groundspeed)
- c) How long will the trip take?



Exit Card!

A light plane is travelling at 175 km/h on a heading of $[N8^\circ E]$ encounters a wind of 40 km/hr from $[N80^\circ E]$. Determine the plane's ground velocity.

Practice on Velocity as Vectors

- Chris is hoping to catch a connecting flight, and time is short. He must cover a distance of 800 m to his next gate in 6 min . Fortunately, a moving walkway extends from gate to gate, going 40 m/min .
 - If the walkway is going in the same direction as he is, how fast must he walk on it to make his flight?
 - If he gets on the walkway and then realizes it is going in the opposite direction, how fast must he walk on it to get to the gate on time?
- Thieves are fleeing in a stolen boat travelling at 30 km/h due west. A police boat is sent to catch them. When the stolen boat is 3 km due north of the police, the police set out at a speed of 40 km/h .
 - In what direction must the police head in order to intercept the thieves?
 - When will the interception occur?
- An airplane which flies at 200 km/h is headed due north. A wind is blowing due east at 40 km/h.
 - What is the magnitude and direction of the plane's velocity relative to the ground?
 - After flying for 90 minutes under these conditions, what is the location of the plane?
- An airplane is flying at 150 km/h at a heading of $W 10^\circ N$. When it lands 2 hours later, its location is 275 km from the starting point, at a heading of $W 20^\circ N$. What is the magnitude and direction of the wind velocity?
- A pilot is planning his flight to an airport which is 400 km southeast of his starting location. His plane flies at 250 km/h but a wind of 20 km/h is blowing from the southwest.
 - What heading should he choose for the plane?
 - How long will the journey take?
- Alex wishes to use a canoe to cross to the other side of a river which is 30 m wide. The river is flowing at 10 m/min and Alex can paddle at 20 m/min. Her goal is a dock which is 6 m downstream from a point directly opposite her starting point. In what direction should she aim her canoe? How long will it take to make this crossing?

Answers

- | | |
|--|--|
| 1. a) 93.3 m/min | b) 173.3 m/min |
| 2. a) [N 48.6° W] | b) 6.8 minutes |
| 3. a) 203.9 km/h [N 11.3° E] | b) 305.9 km [N 11.3° E] from starting point. |
| 4. 28.0 km/h [E 48.5° N] | |
| 5. a) [E 49.6° S] | b) 1 hour, 36 minutes |
| 6. 72° upstream from the bank, 1.6 min | |

3. A river is 2 km wide and flows at 6 km/hr. A motor boat that has a speed of 20 km/hr in still water heads out from one bank perpendicular to the current. A marina lies directly across the river on the opposite bank.
- a) How far downstream from the marina will the boat reach the other bank? [0.6 km downstream]
 - b) How long will it take? [6 minutes]

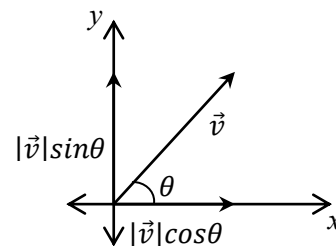
Velocity, Forces, and Cross Product

1. The resultant of \vec{P} and \vec{Q} is a force, \vec{F} is $80N [N70^\circ E]$ and \vec{P} is $25N [E]$. Find the magnitude and direction of \vec{Q} [$57.15N [N61.1^\circ E]$]
2. A particle of mass 11 kg is suspended from a horizontal ceiling by cords from two points A and B on a horizontal ceiling such that $AB = 2 \text{ m}$. The length of the cords are 1.6 m and 1.1 m . Calculate the tension in each cord. [$90.31N$ and $65.1N$]
3. A plane has a velocity of $450 \text{ km/h} [N17^\circ E]$. The wind is 75 km/h from [$N30^\circ W$]. Find the ground velocity of the plane. [$402.6 \text{ km/h} [N24.83^\circ E]$]
4. A force of 25 N makes an angle of 37° with a force of $32N$. Find the magnitude of the equilibrant and the angle it makes with respect to the $32N$ force. [54.1 N 16.15°]
5. If $\vec{a} = [1, 2, 4]$ and $\vec{b} = [2, -3, 1]$ find \hat{n} . [$\hat{n} = \frac{1}{\sqrt{230}}[-10, -9, -7]$]
6. Given $(1, 2, 3)$, $B(3, -1, -2)$, and $C(4, -1, -1)$.
 - a. Find D such that ABCD is a parallelogram. [$D(2, 2, 4)$]
 - b. Find the area of ΔABC . [4.09 units^2]
7. Given $\vec{a} = [1, -1, -2]$, $\vec{b} = [2, -3, -2]$, and $\vec{c} = [3, -2, 4]$. Find the volume of a parallelepiped whose sides are represented by the given vectors. [4 units^3]

CARTESIAN VECTORS

A vector can be identified as a Cartesian Vector if its endpoints can be defined using Cartesian Coordinates.

To write a geometric vector \vec{v} in Cartesian form, you need to use trigonometry. The magnitude of the horizontal component is $|\vec{v}|\cos\theta$, and the magnitude of the vertical component is $|\vec{v}|\sin\theta$, where θ is the angle \vec{v} makes with the horizontal, or the positive x-axis. Thus, $\vec{v} = [|\vec{v}|\cos\theta, |\vec{v}|\sin\theta]$.



Example 1. Write a force of 300N at 30° to the horizontal in Cartesian form.

Example 2. A ship's course is set to travel at 45 km/h, relative to the water, on a heading of 030° . A current of 10 km/h is flowing from a bearing of 140° .

- Write each vector as a Cartesian vector.
- Determine the resultant velocity of the ship.

1. Alice pulls the handle of a wagon with a force of 200N. If the handle forms a 43° angle with the **vertical**, what is the horizontal component of this force?
2. While camping in northern regions at night, people often keep food out of the reach of animals by hanging it between two trees. If a food bag weighing 435 N is tied between two trees 6 m apart by two ropes that are 4 m and 5 m long (after tying) find the tension in each rope.
3. A pilot in an airplane with an airspeed of 625 km/h wishes to fly a city 1500 km due east. There is a wind blowing from [N 25° E] at a speed of 70 km/h.
 - a) In what heading should the pilot steer?
 - b) What will be the groundspeed of the airplane?
 - c) How long will the trip take?
4. A boat crosses a river and arrives at a point directly across from its starting point. The boat can travel at 3.5 m/s and the current is 1.2 m/s. If the river is 450 m wide at the crossing point how long will it take to cross and in what direction must the boat steer?
5. Suppose 2000 J of work is done by pulling a toboggan 260 m by a force applied at an angle of 40° with the horizontal. What is the magnitude of the pulling force?
6. Consider the points $A(1, 0, 2)$, $B(2, 0, 1)$, $C(3, 2, -1)$. If a force of 10N acts in the direction of $[1, 1, -1]$ to move an object from A to B, and distance is measured in meters, how much work is done?
7. An airplane pilot checks her instruments and finds that the speed of the plane relative to the air is 325 km/h. The instruments also show that the plane is pointed in a direction [N 30° W]. A radio report indicates that the wind velocity is 80 km/h blowing from [E 25° N]. What is the velocity of the plane relative to the ground as it is recorded by an air traffic controller in a nearby airport?
8. A large cruise boat is moving at 15 km/h [E 25° S] relative to the water. A person jogging on the ship moves across the ship in a northerly direction at 6 km/h. What is the velocity of the jogger relative to the water?
9. A plane is seen to travel in a direction [N 55° E]. If its ground velocity was 300 km/h and the wind was blowing 50 km/h from [N 45° W], what was the plane's velocity relative to the air?

10. An object weighing 20 kg is suspended by two wires of equal length 50 cm. How far apart must they be attached to the surface above so that the force on each is 150 N?
11. Two vectors, \vec{a} and \vec{b} , of magnitude 3 and 5, respectively, make an angle of 57° with each other. Determine the magnitude and direction of $\vec{b}-\vec{a}$. (Round your answer to one decimal place) [3 marks]