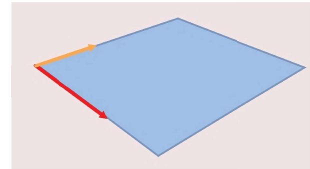
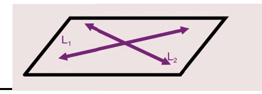
Vector & Parametric Equations of a Plane



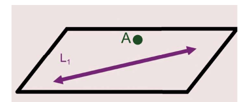
A Plane is:
- a flat surface that extends infinitely far in all directions
- represented by a parallelogram
- denoted by π

How can a plane be formed?

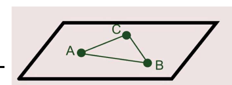
1. by 2 intersecting lines.



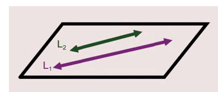
2. by a line and one point not on the line.



3. by 3 non-collinear points.



4. by 2 parallel and distinct lines.



Developing the vector and parametric equations of a plane

Consider: How can we specify an arbitrary point, $P\,(x, y, z)$ on a plane, π ?

Recall: Three non-collinear points A,B, and P_1 determine a plane π .

by triangle law of addition :

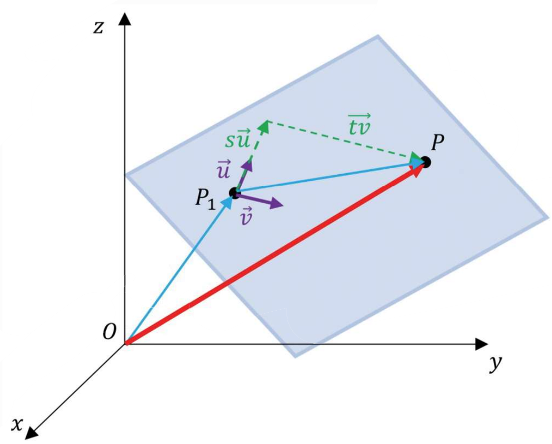
$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P}$$

$$\overrightarrow{P_1P} = s\vec{u} + t\vec{v}$$

$$\therefore \overrightarrow{OP} = \overrightarrow{OP_1} + s\vec{u} + t\vec{v}$$

Position vector of a point on the plane

direction vectors on the plane



Plane in *R*3

Determined by a point P_1 (x_0, y_0, z_0) and **two non collinear direction vectors**

$$\vec{u} = [u_1, u_2, u_3] \text{ and } \vec{v} = [v_1, v_2, v_3]$$

Vector Equation

$$\vec{r} = [x, y, z] = [x_0, y_0, z_0] + s[u_1, u_2, u_3] + t[v_1, v_2, v_3],\ s,t \in \mathbb{R}$$

Parametric Equations of a Plane in *R*3

$x = x_0 + su_1 + tv_1$

$y = y_0 + su_2 + tv_2$

$z = z_0 + su_3 + tv_3$, $s,t \in \mathbb{R}$

Example 1: The plane $\vec{r}=[1,0,0]+t[1,1,1]+s[1,-1,1]; s,t\in R$ contains the point $(3,0,k)$. Find the value of k.

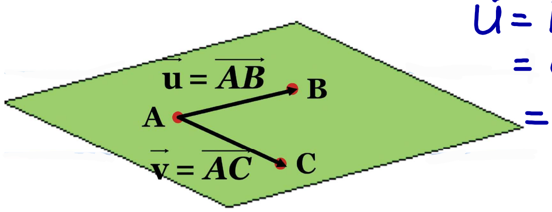
Parametric equations : $x=1+t+s$
$y=t-s$
$z=t+s$

Sub $(3,0,k)$:
$3=1+t+s$
$t+s=2$ — ①
$0=t-s$ — ②
$k=t+s$ — ③

① + ② :
$2t=2$
$\boxed{t=1}$

② + ③
$2t=k$
Sub $t=1$:
$2(1)=k$
$\boxed{k=2}$

Example 2: Find the vector and parametric equations of the plane that contains the points: *A* (1, 0, -3), *B* (2, -3, 1) and *C* (3, 5, -3).



$\vec{u}=\overrightarrow{AB}$
$=\overrightarrow{OB}-\overrightarrow{OA}$
$=[2,-3,1]-[1,0,-3]$
$=[1,-3,4]$

$\vec{v}=\overrightarrow{AC}$
$=[3,5,-3]-[1,0,-3]$
$=[2,5,0]$

$\therefore$ Vector equation is :
$\vec{r}=[1,0,-3]+s[1,-3,4]+t[2,5,0]$, $s,t\in\mathbb{R}$

Parametric equations are : $x=1+s+2t$
$y=-3s+5t$
$z=-3+4s$, $s,t\in\mathbb{R}$

Example 3: What does the following equation represent? Explain.

$$\vec{r}=[-1,2,3]+s[-7,21,14]+t[2,-6,-4]; t,s\in R$$

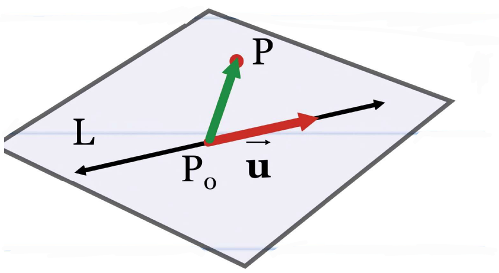
$\vec{u}=[-7,21,14]$
$=-7[1,-3,-2]$

$\vec{v}=[2,-6,-4]$
$=2[1,-3,-2]$

$\therefore \vec{u}=-\frac{7}{2}\vec{v}$

$\vec{u}$ and $\vec{v}$ are collinear vectors, so the equation represents a line in R^3.

Example 4: Determine the vector equation of the plane containing the line $\begin{cases} x = 3 + 5t \\ y = -2 - 2t \\ z = 1 + 3t \end{cases}$ and the point P(1, 2, 3).



Vector equation of the line: $\vec{r} = [3,-2,1] + t[5,-2,3], t \in \mathbb{R}$

$\vec{u} = [5,-2,3]$

$P_o = (3,-2,1)$; $P = (1,2,3)$

$\overrightarrow{P_oP} = \overrightarrow{OP} - \overrightarrow{OP_o}$

$= [1,2,3] - [3,-2,1]$

$= [-2,4,2]$

$= 2[-1,2,1]$

$\vec{v} = [-1,2,1]$

$\therefore$ vector equation of the plane is : $\vec{r} = [1,2,3] + s[-1,2,1] + t[5,-2,3], s,t \in \mathbb{R}$

Practice

1. Find the **vector equation** of the plane that contains line $\frac{3-x}{2} = \frac{y+5}{-2} = \frac{z-4}{3}$ and the y-intercept of $[x,y,z] = [2,1,-4] + t[2,-1,-4], t \in R$.

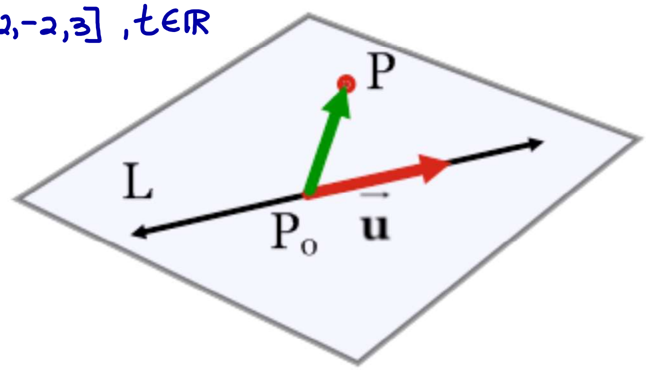
2. Given $L_1 : \vec{r_1} = [3,2,-1] + t[1,2,-1]$ and $L_2 : \vec{r_2} = [1,-4,2] + s[-2,-4,2]$
   (a)Show that the lines are parallel and distinct.
   (b) Find a vector equation of the plane containing both lines

3. Find a vector equation of the plane that passes through the point (6, 0, 0) and contains the line $x = 4 - 2t$, $y = 2 + 3t$, $z = 3 + t$.

Practice-Solution

1. Find the **vector equation** of the plane that contains line $\frac{3-x}{2}=\frac{y+5}{-2}=\frac{z-4}{3}$ and the y-intercept of $[x,y,z]=[2,1,-4]+t[2,-1,-4], t\in R$.

Vector equation of line : $\vec{r}=[3,-5,4]+t[-2,-2,3]$, $t\in R$



$y\text{-int}: x=z=0$

$x=2+2t \rightarrow t=-1$

$y=1-t$

$z=-4-4t \rightarrow t=-1$

$y=1-(-1)$

$=2$

$y\text{-int}=(0,2,0)$

$P_o=(3,-5,4), P=(0,2,0)$

$\overrightarrow{P_oP}=[0,2,0]-[3,-5,4]$

$\vec{u}=[-3,7,-4]$

$\vec{v}=[-2,-2,3]$

$\therefore \vec{r}=[0,2,0]+s[-3,7,-4]+t[-2,-2,3] \quad s,t\in R$

2. Given $L_1:\vec{r}_1=[3,2,-1]+t[1,2,-1]$ and $L_2:\vec{r}_2=[1,-4,2]+s[-2,-4,2]$

(a) Show that the lines are parallel and distinct.

(b) Find a vector equation of the plane containing both lines

a) $\vec{m}_1=[1,2,-1]$

$\vec{m}_2=[-2,-4,2]$

$\vec{m}_2=-2\vec{m}_1$

L_1: $x=3+t$, $y=2+2t$, $z=-1-t$

Sub $(1,-4,2)$ in L_1:

$1=3+t \rightarrow t=-2$

$-4=2+2t \rightarrow t=-3$

$2=-1-t$

Since the t-values are different, the lines are parallel and distinct.

b) $P_o=(3,2,-1), P=(1,-4,2)$

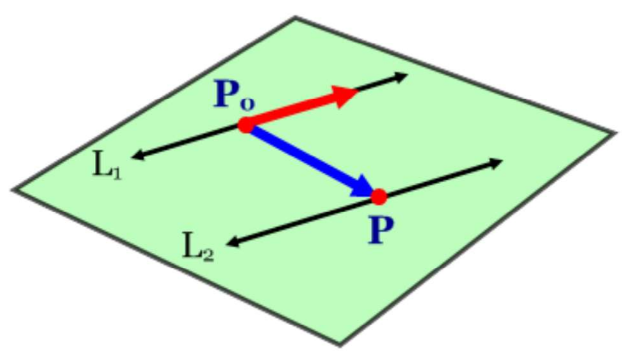
$\overrightarrow{P_oP}=[1,-4,2]-[3,2,-1]$

$=[-2,-6,3]$

$\vec{u}=[-2,-6,3]$

$\vec{v}=[1,2,-1]$

$\vec{r}=[3,2,-1]+q[1,2,-1]+r[-2,-6,3] \quad q,r\in R$



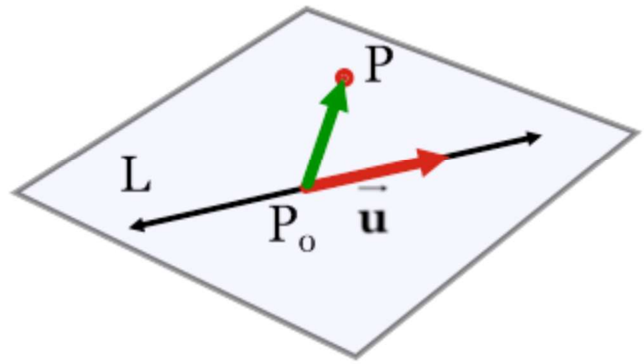
3. Find a vector equation of the plane that passes through the point (6, 0, 0) and contains the line $x = 4 - 2t$, $y = 2 + 3t$, $z = 3 + t$.

$\vec{\mathbf{u}} = [-2, 3, 1]$

$\mathbf{P_0} = (6, 0, 0)\ \&\ \mathbf{P} = (4, 2, 3)$

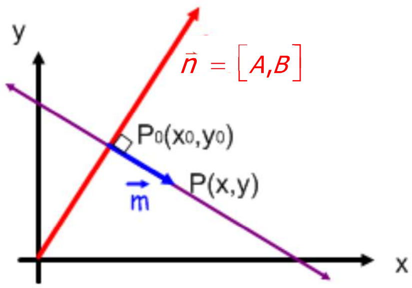
$\overrightarrow{\mathbf{P_0P}} = \vec{\mathbf{v}} = [-2, 2, 3]$

$\vec{\mathbf{r}} = [6, 0, 0] + \mathbf{r}[-2, 3, 1] + \mathbf{q}[-2, 2, 3]\ ;\ \mathbf{r}, \mathbf{q} \in \mathbf{R}$



The Cartesian Equation of a Plane

Recall: For a line:



$\vec{m} = \overrightarrow{P_oP} = [x - x_o, y - y_o]$

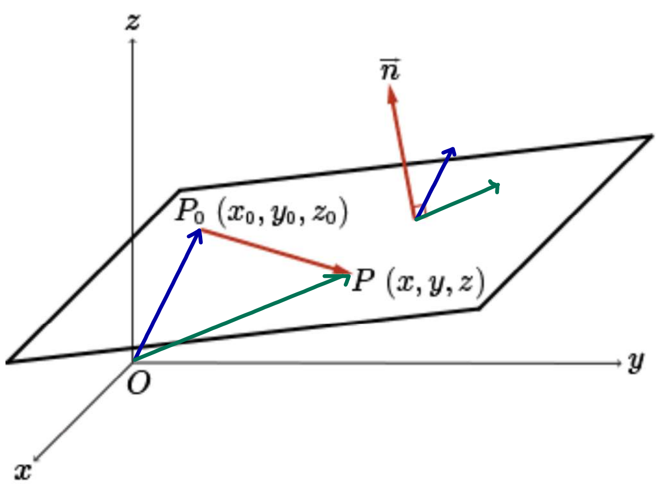
$\vec{n} = [A, B]$

$\therefore \vec{n} \bullet \vec{m} = 0$ (perpendicular)

$[A, B] \bullet [x - x_o, y - y_o] = 0$

$Ax + By + C = 0$ (Cartesian Equation of a Line) in R^2.

For a plane:



If $P_0(x_0, y_0, z_0)$, $P(x, y, z)$ and $\vec{n} = [A, B, C]$ then $\overrightarrow{P_oP} \bullet \vec{n} = 0$, therefore:

$$[x - x_0, y - y_0, z - z_0] \cdot [A, B, C] = 0$$

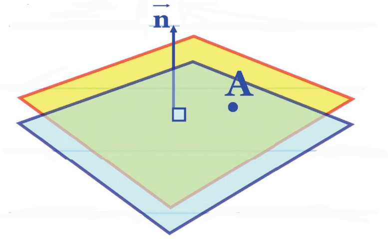
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz + \underbrace{(-Ax_0 - By_0 - Cz_0)}_{D} = 0$$

Scalar Equation of a Plane in R^3

$$\boxed{Ax + By + Cz + D = 0}$$

Ex. 1: Find the equation of π parallel to $2x-3y+4z-3=0$ that contains the point A (1, 3, 5).



$2x-3y+4z-3=0$

$\vec{n}=[2,-3,4]$

Let the equation be $2x-3y+4z+D=0$

Sub $(1,3,5)$:

$2(1)-3(3)+4(5)+D=0$

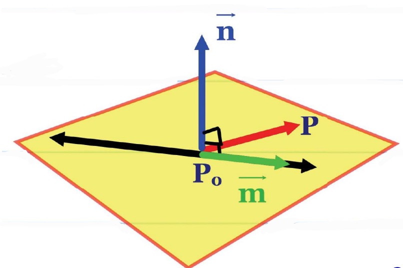
$2-9+20+D=0$

$\boxed{D=-13}$

$\therefore$ Equation of plane π is :

$2x-3y+4z-13=0$

Ex.2: A line has vector equation $\vec{r}=[0,-5,2]+s[1,1,-2]$; $s\in R$ and lies on the plane π. The point P(2,-3,0) also lies on the plane π. Determine the Cartesian equation of this plane.



$P_0(0,-5,2)$, $P(2,-3,0)$

$\vec{P_0P}=[2,-3,0]-[0,-5,2]$

$=[2,2,-2]$

$=2[1,1,-1]$

$\begin{matrix} 1 & -1 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 & -2 \end{matrix}$

$\vec{u}=[1,1,-1]$; $\vec{m}=[1,1,-2]$

$\vec{n}=\vec{u}\times\vec{m}$

$\vec{n}=[-2+1,-1+2,1-1]$

$\vec{n}=[-1,1,0]$

$\therefore$ Cartesian equation is $-x+y+D=0$

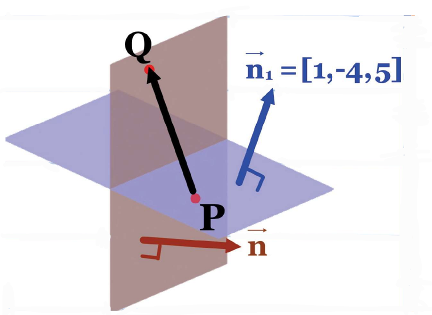
Sub $P(2,-3,0)$: $-2-3+D=0$

$\boxed{D=5}$

$\therefore -x+y+5=0$

or $x-y-5=0$

Ex.3: Determine the scalar equation of the plane that passes through the points P(6, –1, –1) and Q(3, 2, 1) and is perpendicular to the plane $x-4y+5z+5=0$.



$\vec{m}=\vec{PQ}$

$=[3,2,1]-[6,-1,-1]$

$=[-3,3,2]$

Normal to the plane $x-4y+5z+5=0$ is the direction vector of the required plane. $\vec{n_1}=[1,-4,5]$

$\begin{matrix} -3 & 3 & 2 & -3 & 3 & 2 \\ 1 & -4 & 5 & 1 & -4 & 5 \end{matrix}$

$\therefore \vec{n}=\vec{m}\times\vec{n_1}$

$=[-3,3,2]\times[1,-4,5]$

$=[15+8,2+15,12-3]$

$=[23,17,9]$

$\therefore$ Cartesian equation is $23x+17y+9z+D=0$

Sub $P(6,-1,-1)$: $23(6)+17(-1)+9(-1)+D=0$

$112+D=0$

$\boxed{D=-112}$

$\therefore 23x+17y+9z-112=0$

Practice

1. Determine the Cartesian equation of the plane that passes through the point A(6,-1,1), has a z-intercept of -4, and is parallel to the line $\vec{r}=[-2,-1,0]+t[3,3,-1]$.

2. Find scalar equation of a plane contains the line $\begin{cases} x=1+t, \\ y=1-t, \\ z=2t, t\in R \end{cases}$ and is perpendicular to the plane with equation $x+y+z=2$.

3. Find a Cartesian equation of the plane which passes through A(0, 1, -3), and is parallel to the lines

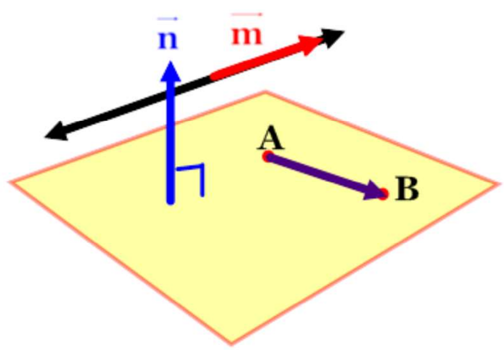
$$L_1: \frac{x-1}{5}=\frac{y+2}{-2}=\frac{z+273}{-1} \quad \text{and} \quad L_1: \frac{x+3}{-7}=\frac{y-1}{2}, z=1$$

4. Find possible values of k such that the line [x, y, z] = [3, 4, 7] + t[k, 1, -2] is parallel to the plane 3kx + ky + z - 6 = 0.

Practice-Solution

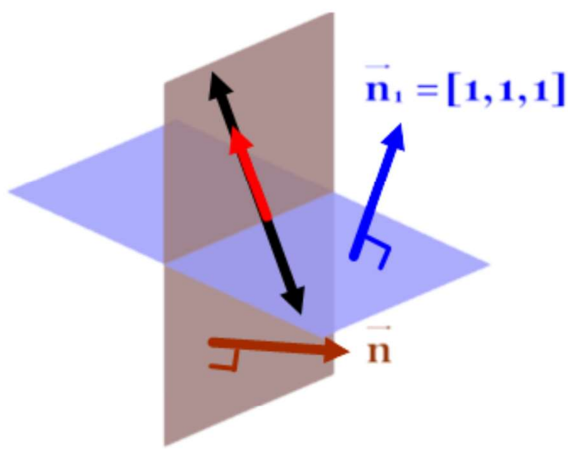
1. Determine the Cartesian equation of the plane that passes through the point A(6,-1,1), has a z-intercept of -4, and is parallel to the line $\vec{r}=[-2,-1,0]+t[3,3,-1]$.

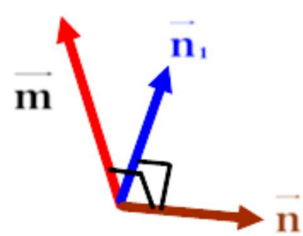
$$\begin{aligned}
&A(6,-1,1)\ \&\ B(0,0,-4)\\
&\overrightarrow{AB}=[-6,1,-5]\\
&\vec{m}=[3,3,-1]\\
&\vec{n}=\overrightarrow{AB}\times\vec{m}=[14,-21,-21]\\
&\qquad\qquad\quad=7[2,-3,-3]\\
&2x-3y-3z+D=0\\
&\text{Sub pt. } B(0,0,-4):\\
&-3(-4)+D=0\\
&D=-12\\
&\therefore \pi: 2x-3y-3z-12=0
\end{aligned}$$



2. Find scalar equation of a plane contains the line $\begin{cases} x=1+t,\\ y=1-t,\\ z=2t, t\in R\end{cases}$ and is perpendicular to the plane with equation $x+y+z=2$.

$$\begin{aligned}
&\vec{m}=[1,-1,2]\\
&\vec{n}_1=[1,1,1]\\
&\vec{n}=\vec{m}\times\vec{n}_1=[-3,1,2]\\
&-3x+y+2z+D=0\\
&\text{Sub pt. } P(1,1,0):\\
&-3(1)+(1)+2(0)+D=0\\
&\qquad\qquad\qquad\qquad D=2\\
&\therefore \pi: -3x+y+2z+2=0
\end{aligned}$$





3. Find a Cartesian equation of the plane which passes through A(0, 1, -3), and is parallel to the lines

$$L_1: \frac{x-1}{5} = \frac{y+2}{-2} = \frac{z+273}{-1} \quad \text{and} \quad L_1: \frac{x+3}{-7} = \frac{y-1}{2}, z = 1$$

$\vec{m}_1 = [5,-2,-1]$

$\vec{m}_2 = [-7,2,0]$

$\vec{n} = \vec{m}_1 \times \vec{m}_2 = [2,7,-4]$

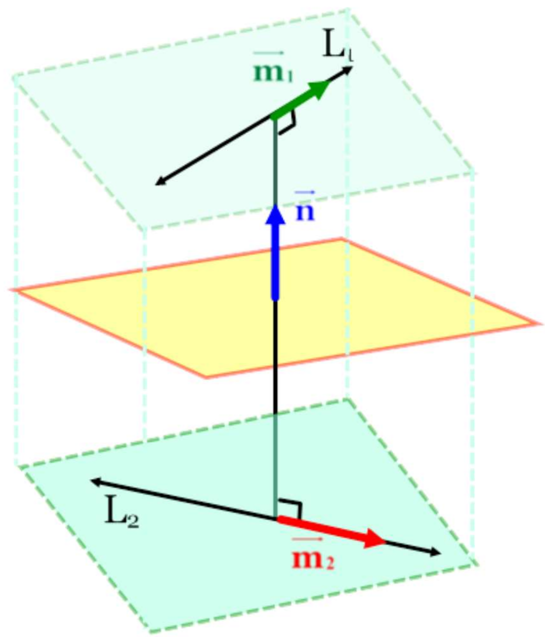
$2x + 7y - 4z + D = 0$

Sub pt. A(0,1,-3):

$2(0) + 7(1) - 4(-3) + D = 0$

$D = -19$

$\therefore \pi: 2x + 7y - 4z - 19 = 0$



4. Find possible values of k such that the line [x, y, z] = [3, 4, 7] + t[k, 1, -2] is parallel to the plane 3kx + ky + z - 6 = 0.

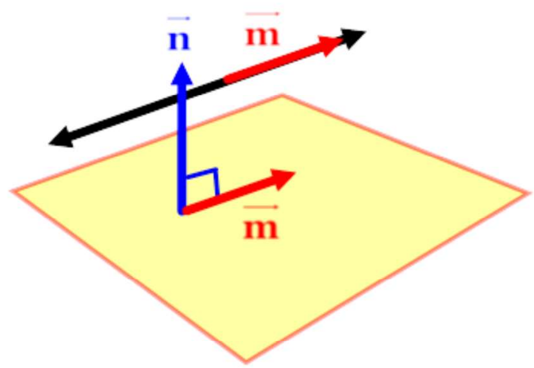
$\vec{n} \bullet \vec{m} = 0$

$[3k, k, 1] \bullet [k, 1, -2] = 0$

$3k^2 + k - 2 = 0$

$(3k-2)(k+1) = 0$

$\boxed{k = -1}, \boxed{k = \frac{2}{3}}$



Unit 5- Exit Card! #1

Name: ______________________ Mark: /11

1. Determine the **parametric equations** of the plane parallel to the line $\frac{x+1}{3}=\frac{2-y}{-4}=\frac{z-5}{6}$ and contains the point $P(2,3,-10)$. ❸

$\vec{u}=[3,4,6]$

P(2,3,-10) & Q(-1,2,5)

$\vec{v}=\overrightarrow{PQ}=[-3,-1,15]$

$\vec{r}=[2,3,-10]+s[3,4,6]+t[-3,-1,15]$

$$\begin{cases} x=2+3s-3t \\ y=3+4s-t \\ z=-10+6s+15t \quad ,t,s\in\mathbb{R} \end{cases}$$

2. Given the planes $\pi_1: -3x+2y-4z-1=0$ and $\vec{r}=[-3,5,2]+s[1,2,-1]+t[2,-4,-3],\ s,t\in\mathbb{R}$, determine the **acute angle**, θ, between the two planes. Round your answer to 1 decimal place. ❸

$\vec{n}_1=[-3,2,-4]$

$\vec{n}_2=[1,2,-1]\times[2,-4,-3]=[-10,1,-8]$

$$\theta=\cos^{-1}\left(\frac{\vec{n}_1\bullet\vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$$

$$\theta=\cos^{-1}\left(\frac{30+2+32}{\left(\sqrt{9+4+16}\right)\left(\sqrt{100+1+64}\right)}\right)$$

$$\theta=\cos^{-1}(0.9252)$$

$$\boxed{\theta\doteq 22.3^{\circ}}$$

3. Determine the coordinates of point *D* such that the points $A(1,1,7)$, $B(2,0,4)$, $C(-5,3,11)$ and $D(k+3, k-1, k-2)$ lie on the same plane. ❺

Equation of a plane contains three points A, B and C:

$$\vec{u} = \overrightarrow{AB} = [1,-1,3]$$
$$\vec{v} = \overrightarrow{AC} = [-6,2,4] = 2[-3,1,2]$$
$$\vec{n} = \vec{u} \times \vec{v} = [1,-1,3] \times [-3,1,2] = [1,7,-2]$$

Cartesian equation of a plane with the normal $\vec{n}$ **is:** $x+7y-2z+D=0$

To find the D value we sub. in point $B(2,0,4)$ **we get:** $2-8+D=0$

$$D=6$$
$$x+7y-2z+6=0$$

Since this plane contains point $D(k+3, k-1, k-2)$**, we get:**

$$k+3+7(k-1)-2(k-2)+6=0$$
$$6k=-6$$
$$\boxed{k=-1}$$

$$\therefore D(2,-2,-3)$$

5-2: Warm-Up

① Find the scalar equation of a plane:

a) Through $A(1,2,-3)$, $B(2,1,5)$ and $C(1,3,-6)$.

b) containing the line $\ell_1: (x,y,z) = (2,1,5) + k(1,-1,3),\ k \in \mathbb{R}$

and parallel to $\ell_2: (x,y,z) = (3,-2,1) + m(2,5,-3),\ m \in \mathbb{R}$.

Warm-Up Solutions

a) $\vec{AB} = [1,-1,8]$
$\vec{AC} = [0,1,-3]$

$$\begin{matrix} 1 & -1 & 8 & 1 & -1 & 8 \\ 0 & 1 & -3 & 0 & 1 & -3 \end{matrix}$$

$$\vec{n} = [3-8,\ 3,\ 1]$$
$$= [-5, 3, 1]$$

$$-5x + 3y + z + D = 0$$

sub in $(1,2,-3)$

$$-5(1) + 3(2) + (-3) + D = 0$$
$$D = 2$$

$\therefore$ the eq'n is $5x - 3y - z - 2 = 0$

b) $\vec{d_1} = [1,-1,3]$
$\vec{d_2} = [2,5,-3]$

$$\begin{matrix} 1 & -1 & 3 & 1 & -1 & 3 \\ 2 & 5 & -3 & 2 & 5 & -3 \end{matrix}$$

$$\vec{n} = [3-15,\ 6+3,\ 5+2]$$
$$= [-12, 9, 7]$$

sub $(2,1,5)$

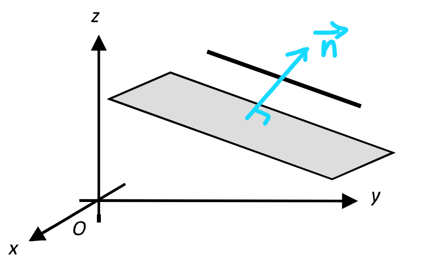
$$-12(2) + 9(1) + 7(5) + D = 0$$

$$-24 + 9 + 35 + D = 0$$
$$D = -20$$

$\therefore$ eq'n is $12x - 9y - 7z + 20 = 0$.

Intersection of a Line and a Plane

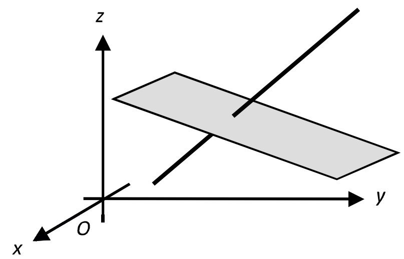
There are three ways for a line to interact with a plane in 3-space.



Line parallel ~~is~~ to the plane

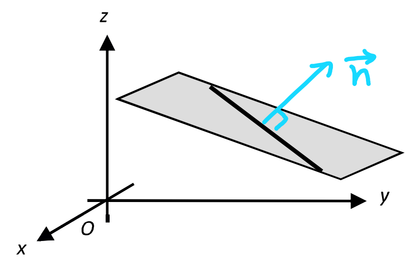
$\boxed{\vec{n}\cdot\vec{m}=0}$

No points in common



Line intersects the plane

Point of intersection



Line lies on the plane

$\boxed{\vec{n}\cdot\vec{m}=0}$

Points in common

Examples: Determine whether the line and plane intersect. If they do, find the point of intersection.

1) $L: \begin{cases} x = 1+2t \\ y = -6+3t \\ z = -5+2t \end{cases}$ $\quad \pi: \; 4x-2y+z-19=0$

$\vec{m} = [2,3,2] \quad \vec{n} = [4,-2,1]$

$\vec{n}\cdot\vec{m}$

$= [2,3,2]\cdot[4,-2,1]$

$= 8-6+2$

$= 4 \neq 0$ $\therefore$ line and plane will intersect.

Sub L into π :

$4(1+2t)-2(-6+3t)+(-5+2t)-19=0$

$4+8t+12-6t-5+2t-19=0$

$4t-8=0$

$\boxed{t=2}$

Sub $t=2$ into L :

$x=1+2(2)$ $\quad y=-6+3(2)$ $\quad z=-5+2(2)$

$\boxed{x=5}$ $\quad \boxed{y=0}$ $\quad \boxed{z=-1}$

$\therefore$ Point of intersection is $(5,0,-1)$.

2) $L: \vec{r} = [0,1,-4] + t[2,-1,1]$ $\quad\quad$ π: $x + 4y + 2z - 4 = 0$

L: $x = 2t$
$y = 1 - t$
$z = -4 + t$
$\vec{m} = [2,-1,1]$

$\vec{n} = [1,4,2]$

$\therefore \vec{n} \cdot \vec{m}$
$= [2,-1,1] \cdot [1,4,2]$
$= 2 - 4 + 2$
$= 0$ $\therefore$ line and plane are parallel.

Sub L into π:

$2t + 4(1-t) + 2(-4+t) - 4 = 0$
$2t + 4 - 4t - 8 + 2t - 4 = 0$
$-8 = 0 \Rightarrow$ false statement $\rightarrow$ No solutions

$\therefore$ line is parallel and distinct to the plane.

3) $L: \begin{cases} x = -4 + 3t \\ y = 0 \\ z = t \end{cases}$ $\quad\quad$ π: $x - 2y - 3z + 4 = 0$

$\vec{n} \cdot \vec{m}$
$= [1,-2,-3] \cdot [3,0,1]$
$= 3 - 0 - 3$
$= 0$

Sub L into π:

$-4 + 3t - 2(0) - 3t + 4 = 0$
$0 = 0$ true statement $\Rightarrow$ infinite solutions

$\therefore$ The line is on the plane. Every point on the line is a point of intersection with the plane.

4) Where does the line $\vec{r} = [6,10,-1] + t[3,4,-1]$ meet the xz-plane?

$x = 6 + 3t$
$y = 10 + 4t$
$z = -1 - t$

xz-plane $\Rightarrow y = 0$
$\therefore 0 = 10 + 4t$
$4t = -10$
$t = -\frac{5}{2}$

$x = 6 + 3\left(-\frac{5}{2}\right)$
$x = \frac{12 - 15}{2}$
$\boxed{x = -\frac{3}{2}}$

$z = -1 + \frac{5}{2}$
$\boxed{z = \frac{3}{2}}$

$\therefore$ The line meet the xz-plane at $\left(-\frac{3}{2}, 0, \frac{3}{2}\right)$.

5-3: Warm-Up

① Find the intersection :

a) $\pi_1: 2x + 5y + 3z = 1$ and $l_2: \frac{x-3}{2} = y+2 = \frac{z}{-3}$

b) $\pi_1: 3x + 5y + 2z = -4$ and $l_2: \frac{x-3}{2} = y+2 = \frac{z}{-3}$

Warm-Up Solutions

1a) $\ell: x = 2t + 3$
$\quad y = t - 2$
$\quad z = -3t$

sub ℓ into π_1:

$$2(2t+3) + 5(t-2) + 3(-3t) = 1$$
$$4t + 6 + 5t - 10 - 9t = 1$$
$$0t = 5$$ false statement

$\therefore \ell$ & π are // & distinct

b) $\ell: x = 2t+3$
$\quad y = t - 2$
$\quad z = -3t$

sub ℓ into π_1:

$$3(2t+3) + 5(t-2) + 2(-3t) = -4$$
$$6t + 9 + 5t - 10 - 6t = -4$$
$$5t = -3$$
$$t = \frac{-3}{5}$$

$x = 2\left(-\frac{3}{5}\right) + 3$
$= -\frac{6}{5} + 3$
$= \frac{9}{5}$

$y = \left(-\frac{3}{5}\right) - 2$
$= -\frac{13}{5}$

$z = -3\left(-\frac{3}{5}\right)$
$= \frac{9}{5}$

$\therefore$ the pt. of intersection is $\left(\frac{9}{5}, -\frac{13}{5}, \frac{9}{5}\right)$.

Distance between Skew Lines in R^3

Recall: $|\mathrm{proj}_{\vec{u}}\vec{v}| = \dfrac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|}$

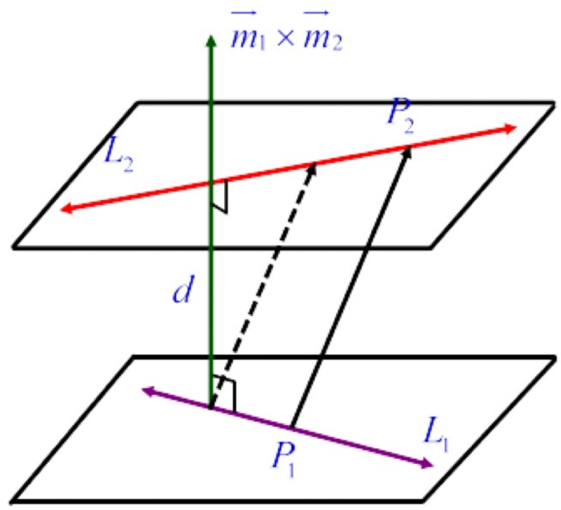
Let P_1 be a point on L_1 with direction vector $\vec{m}_1$ and let P_2 be a point on L_2 with direction vector $\vec{m}_2$ such that L_1 and L_2 are skew lines. The shortest distance between these skew lines is $\boxed{d = \dfrac{|\overrightarrow{P_1P_2} \bullet (\vec{m}_1 \times \vec{m}_2)|}{|\vec{m}_1 \times \vec{m}_2|}}$

Proof:

$$d = |\mathrm{proj}_{\vec{n}}\overrightarrow{P_1P_2}|$$

$$= \frac{|\overrightarrow{P_1P_2}\cdot\vec{n}|}{|\vec{n}|}$$

$$= \frac{|\overrightarrow{P_1P_2}\cdot(\vec{m}_1\times\vec{m}_2)|}{|\vec{m}_1\times\vec{m}_2|}$$



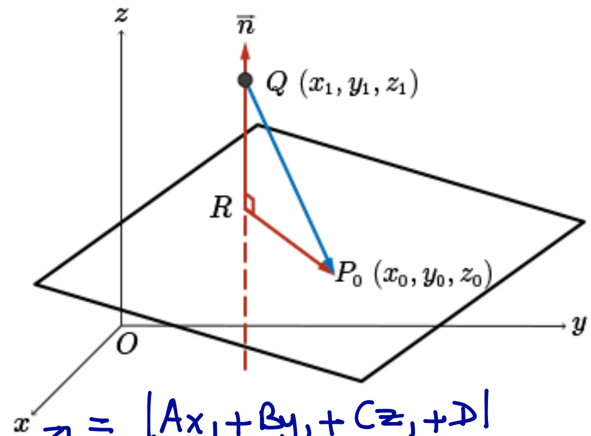
Example : Find the distance between the following skew lines
$[x, y, z] = [-1, 2, -8] + t[-3, 1, 2]$
$[x, y, z] = [2, 5, 11] + s[1, 2, 3]$.

$$\overrightarrow{P_1P} = [2,5,11] - [-1,2,-8]$$
$$= [3,3,19]$$
$$\vec{n} = \vec{d}_1 \times \vec{d}_2$$
$$= [-3,1,2]\times[1,2,3]$$
$$= [3-4, 2+9, -6-1]$$
$$= [-1, 11, -7]$$
$$d = \frac{|\overrightarrow{P_1P_2}\cdot\vec{n}|}{|\vec{n}|}$$
$$= \frac{|[3,3,19]\cdot[-1,11,-7]|}{\sqrt{(-1)^2+11^2+(-7)^2}}$$
$$= \frac{|-3+33-133|}{3\sqrt{19}}$$
$$= \frac{103}{3\sqrt{19}}$$
$$\doteq 7.88 \text{ units}$$

The Distance between a Point and a Plane in R^3

Distance from a Point to a Plane

The distance from a point $Q(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is $\boxed{d = \dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}}$



Proof: $d = |\text{Proj}_{\vec{n}} \overrightarrow{P_0Q}|$

$$= \frac{|\overrightarrow{P_0Q} \cdot \vec{n}|}{|\vec{n}|}$$

$$= \frac{|[x_1 - x_0, y_1 - y_0, z_1 - z_0] \cdot [A, B, C]|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_1 - Ax_0 + By_1 - By_0 + Cz_1 - Cz_0|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_1 + By_1 + Cz_1 - \underbrace{(Ax_0 + By_0 + Cz_0)}_{-D}|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Example: Find the shortest distance between the plane 4x + 2y + z – 16 = 0 and each point.

a) P(10,3,-8)

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|4(10) + 2(3) + 1(-8) - 16|}{\sqrt{4^2 + 2^2 + 1^2}}$$

$$= \frac{22}{\sqrt{21}}$$

b) B(2,2,4)

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|4(2) + 2(2) + 1(4) - 16|}{\sqrt{4^2 + 2^2 + 1^2}}$$

$$= 0$$

Distance between Two Planes

Distance between two parallel planes with equations $\begin{array}{l} Ax + By + Cz + D_1 = 0 \\ Ax + By + Cz + D_2 = 0 \end{array}$ is: $\boxed{d = \dfrac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}}$

Example: Find the shortest distance between the planes with equations 3x – 4y + 5z -10 = 0 and 6x – 8y + 10z – 10 = 0.

$$\pi_1 : 3x - 4y + 5z - 10 = 0 \xrightarrow{\times 2} 6x - 8y + 10z - 20 = 0$$

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \frac{|-20 + 10|}{\sqrt{6^2 + (-8)^2 + 10^2}}$$

$$= \frac{10}{\sqrt{200}}$$

$$= \frac{10}{10\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2} \text{ units}$$

Exit Card!

1) Determine the value(s) of $\boldsymbol{k}$ for which the planes $x - 2y + 2z + k = 0$ and $x - 2y + 2z = k - 2$ have a distance of 8 units.

$D_1 = k \qquad x - 2y + 2z - (k-2) = 0$

$D_2 = -(k-2)$

$\vec{n}_1 = [1, -2, 2]$

$\vec{n}_2 = [1, -2, 2]$

$\therefore \vec{n}_1 = \vec{n}_2$

$\therefore$ Planes are parallel

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

$$8 = \frac{|k + k - 2|}{\sqrt{1^2 + (-2)^2 + 2^2}}$$

$24 = |2k - 2|$

$24 = 2|k-1|$

$12 = |k-1|$

$k - 1 = \pm 12$

$k - 1 = 12$ or $k - 1 = -12$

$\boxed{k = 13}$ $\qquad$ $\boxed{k = -11}$

2) The shortest distance between the skew lines $\begin{matrix}[x,y,z] = [1,1,K] + t[2,-1,\ 1];t \in R \\ [x,y,z] = [0,0,2] + s[3,1,2];s \in R\end{matrix}$ is $\sqrt{35}$ units.

Find the value(s) of $\boldsymbol{K}$.

$\overrightarrow{P_1P_2} = [1, 1, k-2]$

$\vec{n} = [2, -1, 1] \times [3, 1, 2]$

$= [-2-1, 3-4, 2+3]$

$= [-3, -1, 5]$

$d = |\operatorname{Proj}_{\vec{n}} \overrightarrow{P_1P_2}|$

$$= \frac{|\overrightarrow{P_1P_2} \cdot \vec{n}|}{|\vec{n}|}$$

$$\sqrt{35} = \frac{|[1, 1, k-2] \cdot [-3, -1, 5]|}{\sqrt{9 + 1 + 25}}$$

$$\sqrt{35} = \frac{|-3 - 1 + 5k - 10|}{\sqrt{35}}$$

$35 = |5k - 14|$

$5k - 14 = \pm 35$

$5k - 14 = 35$ or $5k - 14 = -35$

$5k = 49 \qquad 5k = -21$

$\boxed{k = \frac{49}{5}}$ $\qquad$ $\boxed{k = \frac{-21}{5}}$

Exit Card #3!

Name:______________________ **Mark: /10**

1. Determine the value(s) of k if the angle between the planes with equation $\sqrt{3}x + z - 5 = 0$ and $kx + 2\sqrt{3}z + 10 = 0$ is $\frac{\pi}{6}$. ③

$$\cos(\theta) = \frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{[\sqrt{3}, 0, 1] \bullet [k, 0, 2\sqrt{3}]}{(2)\left(\sqrt{k^2+12}\right)}$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}k + 2\sqrt{3}}{2\sqrt{k^2+12}}$$

$$\frac{\cancel{\sqrt{3}}}{\cancel{2}} = \frac{\cancel{\sqrt{3}}(k+2)}{\cancel{2}\sqrt{k^2+12}}$$

$$\sqrt{k^2+12} = k+2 \quad (k > -2)$$

$$k^2 + 12 = k^2 + 4k + 4$$

$$8 = 4k$$

$$\boxed{k = 2}$$

2. The distance from $P(3, -3, 1)$ to the plane with equation $Ax - 2y + 6z = 0$ is 3. Determine all possible value(s) of A for which this is true. ③

$$P = (3,-3,1) \; Q = (0,0,0) \quad \& \quad \vec{n} = [A, -2, 6]$$

$$\left|\text{Proj}_{\vec{n}}^{\overrightarrow{PQ}}\right| = \frac{|\overrightarrow{PQ} \bullet \vec{n}|}{|\vec{n}|}$$

$$3 = \frac{\left|[3,-3,1] \bullet [A,-2,6]\right|}{\sqrt{A^2+4+36}}$$

$$3\sqrt{A^2+40} = |3A + 6 + 6|$$

$$\cancel{3}\sqrt{A^2+40} = \cancel{3}|A+4| \; ; \; A \in \mathbb{R}, A \neq -4$$

$$A^2 + 40 = A^2 + 8A + 16$$

$$24 = 8A$$

$$\boxed{A = 3}$$

5-4 Warm-Up

① Find the distance from the point $Q(1,3,-2)$ and the plane

$4x - y - z + 6 = 0$.

② Find the distance between the lines, if possible:

$l_1: \vec{r} = (2,1,0) + t(1,-1,1), t \in \mathbb{R}$

$l_2: \vec{d} = (3,0,-1) + s(2,3,-1), s \in \mathbb{R}$

Warm-Up Solutions

① Pt. on plane: Let $x=1$, $y=1$

$$4-1-z+6=0 \qquad \overrightarrow{PQ}=[0,2,-11]$$
$$z=9$$

$$d=\left|\frac{\overrightarrow{PQ}\cdot\vec{n}}{|\vec{n}|}\right|$$
$$=\frac{\left|[0,2,-11]\cdot[4,-1,-1]\right|}{\sqrt{(4)^2+(-1)^2+(-1)^2}}$$
$$=\frac{9}{3\sqrt{2}} \doteq \frac{3}{\sqrt{2}} \text{ or } 2.12u$$

or $d=\dfrac{|4(1)-(3)-(-2)+6|}{\sqrt{(4)^2+(-1)^2+(-1)^2}}$

$$=\frac{|4-3+2+6|}{\sqrt{18}}$$
$$=\frac{9}{\sqrt{18}} \text{ or } 2.12u$$

②
$$\begin{array}{cccccc} 1 & -1 & 1 & 1 & -1 & 1 \\ 2 & 3 & -1 & 2 & 3 & -1 \end{array}$$

$$\vec{n}=[1-3,\ 2+1,\ 3+2]$$
$$=[-2,3,5]$$

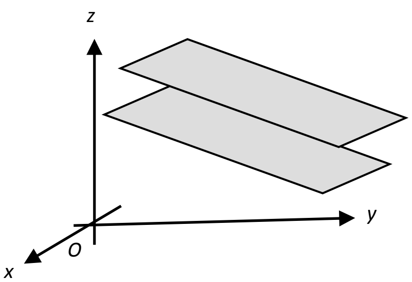
$$\overrightarrow{P_1P_2}=(3,0,-1)-(2,1,0)$$
$$=[1,-1,-1]$$

$$\left|\text{proj}_{\vec{n}}\overrightarrow{P_1P_2}\right|=\left|\frac{\overrightarrow{P_1P_2}\cdot\vec{n}}{|\vec{n}|}\right|$$
$$=\frac{[1,-1,-1]\cdot[-2,3,5]}{\sqrt{(-2)^2+(3)^2+(5)^2}}$$
$$=\frac{|-2-3-5|}{\sqrt{38}}$$
$$=\frac{10}{\sqrt{38}} \doteq 1.62u$$

Intersection of Two Planes

There are three ways for 2 planes to interact in 3-space.

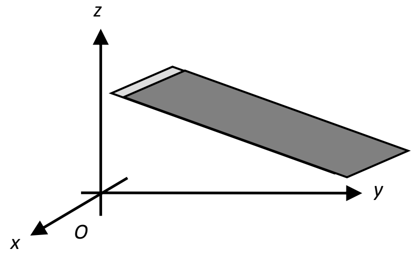
$Ax+By+Cz+D=0$



Parallel and distinct

(no intersection)

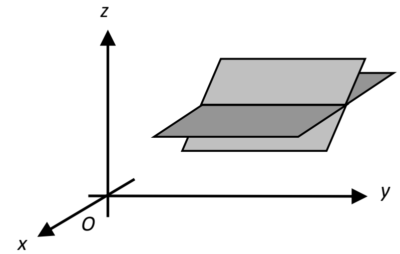
$\boxed{\vec{n_1}=k\vec{n_2} \quad D_1\neq kD_2}$



Parallel and coincident

(intersect at every point)

$\boxed{\begin{matrix}\vec{n_1}=k\vec{n_2}\\ D_1=kD_2\end{matrix}}$



Intersect in a line

$\boxed{\begin{matrix}\vec{n_1}\neq k\vec{n_2}\\ \vec{m}=\vec{n}\times\vec{n_2}\end{matrix}}$

- The planes are parallel and distinct or coincident **iff** (if and only if) the normals are collinear (scalar multiples).
- The planes intersect **iff** the normals are not collinear.

Examples: Investigate the intersection of the planes and find the equation of the line of intersection if applicable.

1) $\pi_1: 4x-5y-2z-1=0$

$\pi_2: x-y-2z=0$

Method ①:

$\vec{n_1}=[4,-5,-2]$

$\vec{n_2}=[1,-1,-2]$

$\begin{matrix}4 & -5 & -2 & 4 & -5 & -2\\ 1 & -1 & -2 & 1 & -1 & -2\end{matrix}$

$\vec{n_1}\neq k\vec{n_2} \Rightarrow$ 2 planes are not parallel

$\vec{m}=\vec{n_1}\times\vec{n_2}$

$\vec{m}=[10-2,-2+8,-4+5]$

$\vec{m}=[8,6,1]$

(Note: can set $y=0$ or $x=0$ or $z=0$)

Set $y=0$ in both equations

$4x-2z=1$ — ①

$x-2z=0$ — ②

① − ②

$3x=1$

$\boxed{x=\frac{1}{3}}$

Sub in ②

$-2z=-\frac{1}{3}$

$\boxed{z=\frac{1}{6}}$

The equation of the line of intersection is

$\vec{r}=[\frac{1}{3},0,\frac{1}{6}]+t[8,6,1]$, $t\in\mathbb{R}$

2) $\pi_1: x+2y+3z-6=0$

$\pi_2: 4x+8y+12z=25$

$\vec{n_1}=[1,2,3]$

$\vec{n_2}=[4,8,12]$

$=4[1,2,3]$

$\vec{n_2}=4\vec{n_1}$

$D_1=-6$

$D_2=-25$

$D_2\neq 4D_1$

∴ The 2 planes are parallel and distinct

Examples: Investigate the intersection of the planes and find the equation of the line of intersection if applicable.

1) $\pi_1 : 4x - 5y - 2z - 1 = 0$
$\pi_2 : x - y - 2z = 0$

Method 2

Step 1: Check for parallel planes (if not specified)

$\vec{n}_1 = [4,-5,-2]$ $\vec{n}_2 = [1,-1,-2]$ $\because \vec{n}_2 \neq k\vec{n}_1$ $\therefore$ 2 planes are <u>not</u> parallel

Step 2: Use elimination to solve:

$$\left.\begin{aligned} 4x-5y-2z-1&=0 \quad (1)\\ x-y-2z&=0 \quad (2)\end{aligned}\right\}$$

2 equations with 3 unknowns
↳ solution will not be unique but a <u>line</u> of intersection involving one parameter.

Eliminate x:
$(1) - 4\times(2)$: $-y+6z-1=0$
$y = 6z-1$

Eliminate y:
$(1) - 5\times(2)$: $-x+8z-1=0$
$x = 8z-1$

Step 3: Assign a variable to be the parameter

Let $z=t$, $t\in\mathbb{R}$ (the parameter)

$\therefore$ The parametric equations of the line of intersection are:

$$x = 8t-1$$
$$y = 6t-1 \quad \text{, where } t\in\mathbb{R}$$
$$z = t$$

Note: Direction vector is $[8,6,1]$ which can be found using $\vec{n}_1 \times \vec{n}_2$

$\vec{n}_1 \times \vec{n}_2 = [4,-5,-2] \times [1,-1,2]$
$= [8,6,1] \leftarrow \vec{d}_1$

(line of intersection of the planes lie in each plane, so $\vec{d}$ is $\perp$ to $\vec{n}_1$ <u>and</u> $\vec{n}_2$)

3) $\pi_1: \begin{cases} x=-4+s \\ y=s+3t \\ z=-s-2t \end{cases}$ $\pi_2: \begin{cases} x=8+6u+2v \\ y=3u+5v \\ z=u-v \end{cases}$

$\vec{r}=[-4,0,0]+s[1,1,-1]+t[0,3,-2], s,t\in\mathbb{R}$

$\vec{r}=[8,0,0]+u[6,3,1]+v[2,5,-1], u,v\in\mathbb{R}$

$\vec{n}=[1,1,-1]\times[0,3,-2]$ $\begin{Bmatrix} 1 & 1 & -1 & 1 & 1 & -1 \\ 0 & 3 & -2 & 0 & 3 & -2 \end{Bmatrix}$

$\vec{n}=[-2+3, 0+2, 3-0]$

$\vec{n}=[1,2,3]$

$\pi_1 \quad x+2y+3z+D=0$

Sub $(-4,0,0)$

$-4+D=0$

$D=4$

$\pi_1 \quad \boxed{x+2y+3z+4=0}$

$\vec{n_2}=[6,3,1]\times[2,5,-1]$ $\begin{Bmatrix} 6 & 3 & 1 & 6 & 3 & 1 \\ 2 & 5 & -1 & 2 & 5 & -1 \end{Bmatrix}$

$\vec{n_2}=[-3-5, 2+6, 30-6]$

$\vec{n_2}=[-8,8,24]$

$\vec{n_2}=-8[1,-1,-3]$ Point $(8,0,0)$

$\pi_2 \quad \boxed{x-y-3z-8=0}$

Method ①:

$\vec{m}=\vec{n_1}\times\vec{n_2}$ $\begin{Bmatrix} 1 & 2 & 3 & 1 & 2 & 3 \\ 1 & -1 & -3 & 1 & -1 & -3 \end{Bmatrix}$

$=[1,2,3]\times[1,-1,-3]$

$=[-6+3, 3+3, -1-2]$

$=[-3, 6, -3]$

$=-3[1,-2,1]$

Sub $z=0$ in π_1 & π_2

$x+2y=-4$ — ①

$x-y=8$ — ②

① − ② $\quad 3y=-12$

$y=-4$

Sub in ①

$x+2(-4)=-4$

$x=-4+8$

$x=4$

$\therefore$ The equation of the line of intersection is:

$\vec{r}=[4,-4,0]+t[1,-2,1], t\in\mathbb{R}$

or

Method ②:

Let $z=t$:

$\pi_1: x+2y+3t+4=0$

$\pi_2: x-y-3t-8=0$

$\pi_1-\pi_2: 3y+6t+12=0$

$y=-2t-4$

$\pi_1+2\cdot\pi_2: 3x-3t-12=0$

$x=t+4$

$\therefore$ The parametric equations of the line are:

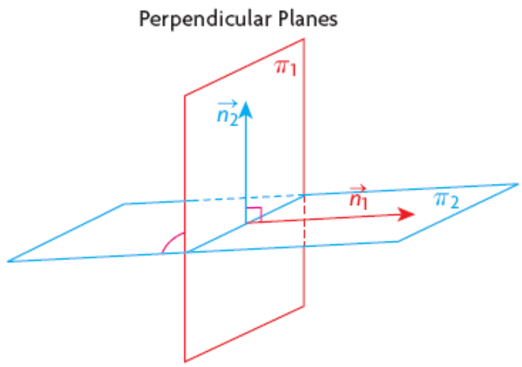
$x=t+4$

$y=-2t-4$, $t\in\mathbb{R}$

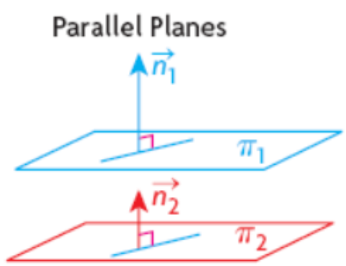
$z=t$

Special Cases

Two planes are perpendicular iff $\vec{n}_1 \cdot \vec{n}_2 = 0$

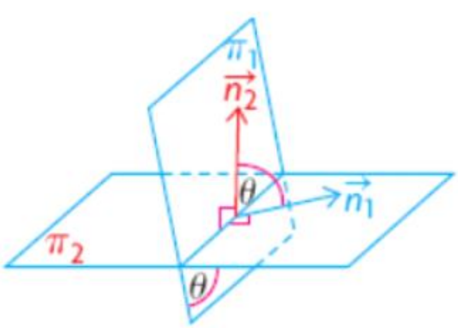


Two planes are parallel iff $\vec{n}_1 = k\vec{n}_2$



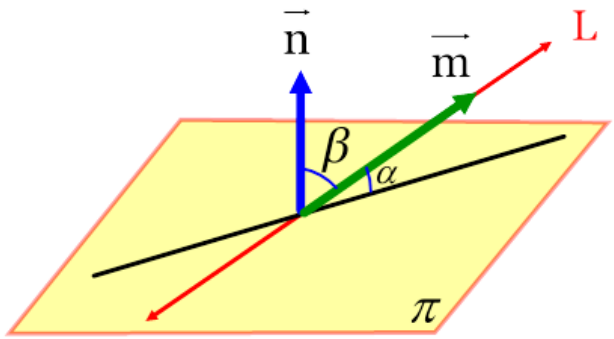
Angle between two planes:

The angle between two planes is the same as angle between their normals!



$$\boxed{\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}}$$

Angle between a line and a plane



$$\cos\beta = \frac{\vec{n} \bullet \vec{m}}{|\vec{n}||\vec{m}|}$$

Ex. Line ℓ with equation $\vec{r} = [5,-1,4] + t[2,-2,0]$, $t \in R$ intersects the plane $\pi: \mathbf{a}x + z = 5\mathbf{a} + 4$ at an angle of $\frac{\pi}{6}$. Find the value of **a**, where **a** is a positive constant

$\vec{n} = [a,0,1]$

Ex. $\alpha = \frac{\pi}{6} \Rightarrow \beta = \frac{\pi}{3}$

$\cos\beta = \frac{\vec{n}\cdot\vec{m}}{|\vec{n}||\vec{m}|}$

$\cos\left(\frac{\pi}{3}\right) = \frac{[a,0,1]\cdot[2,-2,0]}{\left(\sqrt{a^2+1}\right)\left(\sqrt{2^2+(-2)^2}\right)}$

$\frac{1}{2} = \frac{2a}{\sqrt{8(a^2+1)}}$

$4a = \sqrt{8(a^2+1)}$

$16a^2 = 8a^2 + 8$

$8a^2 = 8$

$a^2 = 1$

$\boxed{a=1}$, $a > 0$

Practice

1. Determine the intersection of the planes π_1: $6x - 3y + 12z = 9$ and π_2: $4x - 2y + 8z = 6$ if it exists. If it does, determine the acute angle between the two planes.

2. Determine the intersection of the planes $\pi 1$: $x + 3y + z = 2$ and $\pi 2$: $2x + y - z = -1$ if it exists. If it does, determine the acute angle between the two planes.

3. Find the intersection of π_1: $2x + y + 3z = -7$ and π_2 : $x - y + z = -5$ by using the cross product of the two normal vectors to find a direction vector of the line. When finding your point, set x=0 in both equations and solve for y and z.

4. The planes with equations π_1: $3x - 5y + 2z = 1$ and π_2: $3x - 5y + 2z = k$ are parallel $k \epsilon \boldsymbol{R}$.
a. A line intersects π_1 perpendicularly at the point (2, 1, 0), where does it intersect π_2?
b. Find all values of k for which the distance between the planes equals 3.

Practice

1. Determine the intersection of the planes π_1: $6x - 3y + 12z = 9$ and π_2: $4x - 2y + 8z = 6$ if it exists. If it does, determine the acute angle between the two planes.

$$\vec{n}_1 = [6,-3,12]\ ,\ D_1 = -9$$
$$\vec{n}_2 = [4,-2,8]\ ,\ D_2 = 6$$
$$\vec{n}_2 = \frac{3}{2}\vec{n}_1 \quad \text{and} \quad D_2 = \frac{3}{2}D_1$$

$\therefore$ **Two planes are coinsident**

2. Determine the intersection of the planes $\pi 1$: $x + 3y + z = 2$ and $\pi 2$: $2x + y - z = -1$ if it exists. If it does, determine the acute angle between the two planes.

$$\vec{n}_1 = [1,3,1]\ ,\quad D_1 = -2$$
$$\vec{n}_2 = [2,1,-1]\ ,\ D_2 = 1$$
$$\vec{m} = \vec{n}_1 \times \vec{n}_2$$
$$= [-4,3,-5]$$
$$\text{Let } y = 0: x + z = 2 \quad (1)$$
$$2x - z = -1 \quad (2)$$
$$(1)+(2): 3x = 1$$
$$x = \frac{1}{3}\ \&\ z = \frac{5}{3}$$
$$L: \vec{r} = \left[\frac{1}{3}, 0, \frac{5}{3}\right] + t[-4,3,-5]\ , t \in \mathbb{R}$$

$$\cos(\theta) = \frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$
$$= \frac{[1,3,1]\bullet[2,1,-1]}{\left(\sqrt{1^2+3^2+1^2}\right)\left(\sqrt{2^2+1^2+1^2}\right)}$$
$$= \frac{4}{\sqrt{66}}$$
$$\theta \doteq 60.5^\circ$$

3. Find the intersection of π_1: $2x + y + 3z = -7$ and π_2 :$x - y + z = -5$ by using the cross product of the two normal vectors to find a direction vector of the line. When finding your point, set x=0 in both equations and solve for y and z.

$$\vec{n}_1 = [2,1,3]$$
$$\vec{n}_2 = [1,-1,1]$$

$$\begin{matrix} 2 & 1 & 3 & 2 & 1 & 3 \\ 1 & -1 & 1 & 1 & -1 & 1 \end{matrix}$$

$$\vec{m} = \vec{n}_1 \times \vec{n}_2$$
$$\vec{m} = [1+3, 3-2, -2-1]$$
$$\vec{m} = [4,1,-3]$$

Sub $x=0$ in π_1 + π_2

$$y + 3z = -7 \quad — \ 1$$
$$-y + z = -5 \quad — \ 2$$

1 + 2

$$4z = -12$$
$$\boxed{z = -3}$$

Sub in 1

$$y + 3(-3) = -7$$
$$y = -7 + 9$$
$$\boxed{y = 2}$$

The equation of the line is

$$\vec{r} = [0,2,-3] + t[4,1,-3],\ t \in \mathbb{R}$$

4. The planes with equations π_1: $3x - 5y + 2z = 1$ and π_2: $3x - 5y + 2z = k$ are parallel $k \in \mathbf{R}$.

a. A line intersects π_1 perpendicularly at the point (2, 1, 0), where does it intersect π_2?

b. Find all values of k for which the distance between the planes equals 3.

a) $\vec{n}_1 = [3,-5,2]$ $\quad \vec{n}_2 = [3,-5,2]$

$\vec{n}_1 = \vec{n}_2$ The planes are parallel

Equation of line is $\vec{r} = [2,1,0] + t[3,-5,2]$

$$x = 2+3t$$
$$y = 1-5t$$
$$z = 2t$$

Sub in π_2 $\quad 3(2+3t) - 5(1-5t) + 2(2t) = k$

$$6+9t-5+25t+4t = k$$
$$38t+1 = k$$
$$t = \frac{k-1}{38}$$

Sub in parametric equations

$$x = 2+3\left(\frac{k-1}{38}\right) \qquad y = 1-5\left(\frac{k-1}{38}\right) \qquad z = 2\left(\frac{k-1}{38}\right)$$

$$x = 2+\frac{3k-3}{38} \qquad y = \frac{38-5k+5}{38} \qquad z = \frac{k-1}{19}$$

$$x = \frac{76+3k-3}{38} \qquad y = \frac{43-5k}{38}$$

$$x = \frac{73+3k}{38}$$

$\therefore$ It will intersect π_2 at $\left(\frac{73+3k}{38}, \frac{43-5k}{38}, \frac{k-1}{19}\right)$, $k \in \mathbb{R}$

b) $d = \frac{|D_1 - D_2|}{\sqrt{A^2+B^2+C^2}}$

$$3 = \frac{|-1+k|}{\sqrt{3^2+(-5)^2+2^2}}$$

$$\pm 3 = \frac{-1+k}{\sqrt{38}}$$

$$\pm 3\sqrt{38} + 1 = k$$

$k = 1+3\sqrt{38}$ or $k = -1+3\sqrt{38}$

Name: ____________________________ **Exit Card #2** Mark: /10

Part A. Multiple Choice. Chose the best choice for questions 1-3

C ___ 1. What value of *k* will make the planes $\pi_1 : 2x - ky + 3z - 1 = 0$ and $\pi_2 : 2kx + 3y - 2z = 4$ perpendicular? ①

A. 2 B. 4 C. 6 D. 8

D ___ 2. Which of the following is a line parallel to the line of intersection of two planes given by equations: $-3x + y - z = -2$ and $5x - 2y - 3z = -9$?①

A. $x = -s + 12, y = -4s + 3, z = 2s$
C. $x = s + 12, y = -12s, z = 3$
B. $x = -4s + 12, y = -13s + 34, z = s$
D. $x = -5s + 10, y = -14s + 3, z = s + 5$

D ___ 3. Determine the value of *k* for which the planes $x - 2y + kz + 7 = 0$ and $x - 2y - 2 = 0$ have an angle of intersection of 60°. ①

A. 3 B. $\pm\sqrt{10}$ C. ± 4 D. $\pm\sqrt{15}$

4. Given the two planes below, find the value(s) of **k** (if there is any) that make the two planes intersect in the desired way, if possible. Explain your reasoning.

$$6x - 9y + 15z = 21$$
$$10x - 15y + \mathbf{k}z = 35$$

a) Along a whole plane ①

Let $\pi_1 : 6x - 9y + 15z - 21 = 0$, divide by 3, we get $\pi_1 : 2x - 3y + 5z - 7 = 0$.

Let $\pi_2 : 10x - 15y + kz - 35 = 0$, divide by 5, we get $\pi_2 : 2x - 3y + (\frac{k}{5})z - 7 = 0$.

For π_1 and π_2 to intersect in a plane, they have to be coincident, or more precisely, the same plane. Looking at the equations, when $5 = \frac{k}{5}$, or $\boxed{k = 25}$, the two planes will have the same equation and hence will be coincident.

b) Along a single line. ①

If $k \neq 25$, then the normal of π_1 is not parallel to the normal of π_2 and therefore, two planes must intersect in a line.

c) No intersection ①

The only way π_1 and π_2 can have no intersection is if they are parallel and distinct. However, from (a), we see that if they are parallel (same normal vectors) they will also have the same equation, and will be coincident. Therefore, there is no value of k for which the two planes will be parallel and distinct.

5. Suppose ℓ is the line of intersection of the two planes $\pi_1: 2x-3y+4z=3$ and $\pi_2: 2x+3y-2z=-3$. Determine the point on ℓ that is closest to the point $P(2,1,-2)$.④

$\pi_1: 2x-3y+4z=3$

$\pi_2: 2x+3y-2z=-3$

Let $\vec{m}$ be the direction vector of the line of intersection.

$\vec{m}=[2,-3,4]\times[2,3,-2]$

$=[-6,12,12]$

$=6[-1,2,2]$

set $\boxed{z=0}: 2x-3y=3$

$\underline{2x+3y=-3}$

$\oplus\ 4x=0$

$\boxed{x=0}$ and $\boxed{y=-1}$

$\boxed{\text{Equation of the line of intersection}: \vec{r}=[0,-1,0]+s[-1,2,2]}$

Let point $A(-s,-1+2s,2s)$ is the point lies on the line that has closest distance to point P(2,1,-2),

since $\overrightarrow{AP}\perp\vec{m}$, we have $\overrightarrow{AP}\bullet\vec{m}=0$

$\overrightarrow{AP}=[2+s,2-2s,-2-2s]$

$[2+s,2-2s,-2-2s]\bullet[-1,2,2]=0$

$-2-s+4-4s-4-4s=0$

$9s=-2$

$s=\frac{-2}{9}$

$\boxed{\therefore A\left(\frac{-2}{9},\frac{-13}{9},\frac{-4}{9}\right)}$

1. Given the line $\vec{r} = [12,-8,-4] + t[-3,4,2]$, find the intersection(s) with the xy- plane.

2. Find vector and parametric equations of the plane that contains the two intersecting lines ,

$\vec{r} = [3,-1,2] + s[4,0,1]$ and , $\vec{r} = [3,-1,2] + t[4,0,2]$.

3. Find a scalar equation of the plane that contains the origin and the point (2, -3, 2) and is perpendicular to the plane $x + 2y - z + 3 = 0$.

4. Determine the parametric equations for the plane containing the 2 parallel lines:

$L_1 : \vec{r} = [0,1,3] + t[-6,-3,6]$ and $L_2 : \vec{r} = [-4,5,-4] + s[4,2,-4]$.

5. Determine the **scalar** equation of the plane parallel to the plane $\pi_1 : 3x + y - 2z - 4 = 0$ and containing the point of intersection of lines $L_1 : x = 7 + 2s, y = 2 + s, z = -6 - 3s$ and

$L_2 : \vec{r} = [3,9,13] + t[1,5,5]$.

6. Find the scalar equation of the plane that is perpendicular to the plane $\vec{r} = [4,-5,2] + s[2,1,3] + t[-1,4,0]$ and intersects it at the line $\vec{r} = [4,-5,2] + t[1,-1,1]$.

7. Find the **vector** equation of the line of intersection between the plane $5x - 2y + 3z - 2 = 0$ and the *xy*-plane.

8. Consider the lines $l_1 : \vec{r} = [1,-2,4] + s[1,1,-3]$ and $l_2 : \vec{r} = [4,-2,k] + t[2,3,1]$ Determine the equation of the plane that contains l_1 and is parallel to l_2 .

9. Find the equation of the plane that contains the line $\vec{r} = [3,1,0] + t[2,1,4]$ and is perpendicular to the plane $\pi : \vec{p} = [1,1,1] + k[1,0,5] + s[-4,2,3]$.

10. Determine the measure of the acute angle to the nearest degree between the lines:

$l_1 : \vec{r} = [1,-2,3] + t[4,1,-1]$ and $l_2 :\ x = 3 - t\ ,\ y = t\ , z = 3 + 2t$.

11. Determine the Cartesian equation of the plane that contains the point (2,-1,1) and is perpendicular to the line joining points (-1,3,2) and (4,0,-2).

12. Determine the value of k if the acute angle between the planes 2x+ky-8=0 and x-3y+z+4=0 is 60° .

1. Given the line $\vec{r}=[12,-8,-4]+t[-3,4,2]$, find the intersection(s) with the xy plane.

Equation of xy- plane: z=0

$$x = 12 - 3t$$
$$y = -8 + 4t$$
$$z = -4 + 2t \xrightarrow{z=0} t = 2$$
$$\text{P.O.I} : (6,0,0)$$

2. Find vector and parametric equations of the plane that contains the two intersecting lines ,

$\vec{r}=[3,-1,2]+s[4,0,1]$ and , $\vec{r}=[3,-1,2]+t[4,0,2]$.

$$\vec{r} = [3,-1,2] + p[4,0,1] + q[2,0,1] \ ; p,q \in R$$
$$x = 3 + 4p + 2q$$
$$y = -1$$
$$z = 2 + p + q$$

3. Find a scalar equation of the plane that contains the origin and the point (2, -3, 2) and is perpendicular to the plane $x + 2y - z + 3 = 0$.

$$\vec{n_1} = [1,2,-1]$$
$$\overrightarrow{OP} = [2,-3,2]$$
$$\vec{n_2} = \vec{n_1} \times \overrightarrow{OP}$$
$$= [1,2,-1] \times [2,-3,2]$$
$$= [1,-4,-7]$$
$$\text{equation of plane} : x - 4y - 7z = 0$$

4. Determine the parametric equations for the plane containing the 2 parallel lines:

$L_1 : \vec{r}=[0,1,3]+t[-6,-3,6]$ and $L_2 : \vec{r}=[-4,5,-4]+s[4,2,-4]$.

$$\vec{u} = \overrightarrow{PQ} = [0,1,3] - [-4,5,-4] = [4,-4,7]$$
$$\vec{m_1} = -3[2,1,-2] \ , \ \vec{m_2} = 2[2,1,-2]$$
$$\vec{v} = [2,1,-2]$$
$$\text{equation of plane} : \ x = 4s + 2t$$
$$y = 1 - 4s + t$$
$$z = 3 + 7s - 2t$$

5. Determine the **scalar** equation of the plane parallel to the plane $\pi_1: 3x + y - 2z - 4 = 0$ and containing the point of intersection of lines $L_1: x = 7 + 2s, y = 2 + s, z = -6 - 3s$ and $L_2: \vec{r} = [3,9,13] + t[1,5,5]$.

$L_1: x = 7 + 2s$ $\quad$ $L_2: x = 3 + t$

$y = 2 + s$ $\quad$ $y = 9 + 5t$

$z = -6 - 3s$ $\quad$ $z = 13 + 5t$

$7 + 2s = 3 + t \longrightarrow 2s - t = -4 \quad (1)$

$2 + s = 9 + 5t \longrightarrow s - 5t = 7 \quad (2)$

$-6 - 3s = 13 + 5t \longrightarrow 3s + 5t = -19 \quad (3)$

From (2) & (3) : $s = -3$, $t = -2$

Check: $2s - t = -4$

$2(-3) - (-2) = 4$

$\boxed{\text{P.O.I is } (1,-1,3)}$

$\pi: 3x + y - 2z + D = 0 \quad \downarrow \text{point}(1,-1,3)$

$3 - 1 - 6 + D = 0$

$D = 4$

$\boxed{\therefore \pi: 3x + y - 2z + 4 = 0}$

6. Find the scalar equation of the plane that is perpendicular to the plane $\vec{r}=[4,-5,2]+s[2,1,3]+t[-1,4,0]$ and intersects it at the line $\vec{r}=[4,-5,2]+t[1,-1,1]$.

$\vec{n}_1 = [2,1,3] \times [-1,4,0] = [-12,-3,9] = -3[4,1,-3]$

$\vec{m} = [1,-1,1]$

$\vec{n} = \vec{n}_1 \times \vec{m}$

$= [4,1,-3] \times [1,-1,1]$

$= [-2,-7,-5]$

$= -[2,7,5]$

$\pi: 2x + 7y + 5z + D = 0 \quad \downarrow (4,-5,2)$

$8 - 35 + 10 + D = 0$

$D = -17$

$\boxed{\therefore \pi: 2x + 7y + 5z - 17 = 0}$

7. Find the **vector** equation of the line of intersection between the plane $5x - 2y + 3z - 2 = 0$ and the *xy*-plane.

***Equation of xy*-plane : z=0**

Scalar equation of line of intersection between two planes: $5x - 2y - 2 = 0$

$\vec{m} = [2,5]$

$\vec{r} = [0,-1] + t[2,5]$

8. Consider the lines $l_1: \vec{r} = [1,-2,4] + s[1,1,-3]$ and $l_2: \vec{r} = [4,-2,k] + t[2,3,1]$ Determine the equation of the plane that contains l_1 and is parallel to l_2 .

$\vec{n} = [1,1,-3] \times [2,3,1] = [10,-7,1]$

$10x - 7y + z + D = 0 \leftarrow (1,-2,4)$

$10 + 14 + 4 + D = 0$

$D = 28$

$\boxed{\therefore \ \pi : 10x - 7y + z + 28 = 0}$

9. Find the equation of the plane that contains the line $\vec{r} = [3,1,0] + t[2,1,4]$ and is perpendicular to the plane $\pi : \vec{p} = [1,1,1] + k[1,0,5] + s[-4,2,3]$.

$\vec{n}_1 = [1,0,5] \times [-4,2,3] = [-10,-23,2]$

$\vec{m} = [2,1,4]$

$\vec{n} = \vec{n}_1 \times \vec{m}$

$= [-10,-23,2] \times [2,1,4]$

$= [-94,44,36]$

$= 2[-47,22,18]$

$\pi : -47x + 22y + 18z + D = 0 \ \downarrow (3,1,0)$

$-141 + 22 + D = 0$

$D = -119$

$\boxed{\therefore \pi : -47x + 22y + 18z - 119 = 0}$

10. Determine the measure of the acute angle to the nearest degree between the lines:

$l_1 : \vec{r} = [1,-2,3] + t[4,1,-1]$ and $l_2 : \ x = 3 - t \ , \ y = t \ , z = 3 + 2t$.

$$\cos\theta = \frac{\vec{m}_1 \bullet \vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$$

$$= \frac{[4,1,-1] \bullet [-1,1,2]}{\left(\sqrt{16+1+1}\right)\left(\sqrt{1+1+4}\right)}$$

$$\cos\theta = \frac{-5}{\sqrt{108}}$$

$\boxed{\theta \doteq 118.7^\circ}$

11. Determine the Cartesian equation of the plane that contains the point **A**(2,-1,1) and is perpendicular to the line joining points **B**(-1,3,2) and **C**(4,0,-2).

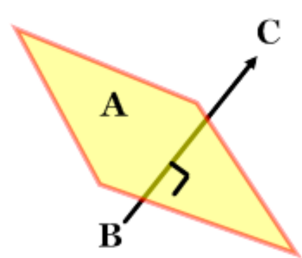
$\vec{n} = \vec{BC} = [5,-3,-4]$

$\pi : 5x - 3y - 4z + D = 0$

$A \in \pi : 10 + 3 - 4 + D = 0$

$D = -9$

$\boxed{\pi : 5x - 3y - 4z - 9 = 0}$



12. Determine the value(s) of k if the acute angle between the planes 2x+ky-8=0 and x-3y+z+4=0 is 60°.

$$\cos\theta = \frac{\vec{n_1}\bullet\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$\frac{1}{2} = \frac{[2,k,0]\bullet[1,-3,1]}{\left(\sqrt{4+k^2}\right)\left(\sqrt{1+9+1}\right)}$$

$$\frac{1}{2} = \frac{2-3k}{\sqrt{11(4+k^2)}} \qquad (2-3k>0 \text{ or } k<\frac{2}{3})$$

$$11(4+k^2) = 4(2-3k)^2$$

$$44+11k^2 = 16-48k+36k^2$$

$$0 = 25k^2-48k-28$$

$$k = \frac{24\pm2\sqrt{319}}{25}$$

$$k \doteq 2.39 \text{ or } \boxed{k \doteq -0.47}$$

5-5 Warm-Up:

① Investigate the intersection of:

$$\pi_1: 2x-5y+3z=12 \text{ and } \pi_2: 3x+4y-3z=6.$$

② Find the parametric form of the line of intersection of the planes π_1 and π_2.

$\pi_1: x=-4+s$
$y=s+3t$
$z=-s-2t, \; t,s\in\mathbb{R}$

$\pi_2: x=8+6u+2v$
$y=3u+5v$
$z=u-v, \quad u,v\in\mathbb{R}$

Warm-Up Solutions

Method 1:

① $\vec{n}_1 = [2, -5, 3]$ $\qquad \vec{n}_1 \neq k\vec{n}_2 \quad \therefore \pi_1 \not\parallel \pi_2$

$\vec{n}_2 = [3, 4, -3]$

Eliminate x:

$\pi_1 \times 3$: $6x - 15y + 9z = 36$

$\pi_2 \times 2$: $\ominus\, 6x + 8y - 6z = 12$

$-23y + 15z = 24$

$y = \frac{15}{23}z - \frac{24}{23}$

Eliminate y:

$\pi_1 \times 4$: $8x - 20y + 12z = 48$

$\pi_2 \times 5$: $\oplus\, 15x + 20y - 15z = 30$

$23x \quad -3z = 78$

$x = \frac{3}{23}z + \frac{78}{23}$

$\therefore$ Let $z = t$

$x = \frac{3}{23}t + \frac{78}{23}$

$y = \frac{15}{23}t - \frac{24}{23}, t \in \mathbb{R}$

Let $x = t$

$y = 5t - 18$

$z = \frac{23}{3}x - 26$

Let $y = t$

$x = \frac{1}{5}t + \frac{18}{5}$

$z = \frac{23}{15}t + \frac{24}{15}$

Method 2:

Vector that is parallel to both planes

$\vec{d} = [2, -5, 3] \times [3, 4, -3]$

$= [3, 15, 23]$

Find the common point on both planes

Let $x = 0$

π_1: $2(0) - 5y + 3z = 12$

$-5y + 3z = 12$ ①

π_2: $3(0) + 4y - 3z = 6$

$4y - 3z = 6$ ②

$4 \times$ ① $-20y + 12z = 48$

$5 \times$ ② $\oplus\, 20y - 15z = 30$

$-3z = 78$

$z = -26$

sub into ①:

$-5y + 3(-26) = 12$

$-5y = 90$

$y = -18$

pt. is $(0, -18, -26)$

$\therefore$ the equation of the line of int.

is $(x, y, z) = (0, -18, -26) + t(3, 15, 23), t \in \mathbb{R}$

② Find scalar equations of π_1 & π_2:

$\vec{n}_1 = [1,1,-1] \times [0,3,-2]$
$= [1,2,3]$

$\vec{n}_2 = [6,3,1] \times [2,5,-1]$
$= [-8,8,24]$

$\because \vec{n}_1 \neq k\vec{n}_2 \therefore \pi_1 \not\parallel \pi_2$

pt $(-4,0,0)$

pt $(8,0,0)$

$SE_{\pi_1}: (x+4, y, z)\cdot(1,2,3)=0$

$SE_{\pi_2}: (x-8, y, z)\cdot(8,8,24)=0$

$\pi_1: x+2y+3z+4=0$ ①

$\pi_2: 8x-8y-24z-64=0$ ②

Eliminate x:

①×8: $8x+16y+24z+32=0$
② : ⊖ $\underline{8x-8y-24z-64=0}$
$24y+48z+96=0$
$24y=-48z-96$
$y=-2z-4$

Eliminate y:

①×4: $4x+8y+12z+16=0$
② : ⊕ $\underline{8x-8y-24z-64=0}$
$12x \quad -12z-48=0$

$12x=12z+48$
$x=z+4$

$\therefore$ Let $z=t$
$x=t+4$
$y=-2t-4$, $t\in\mathbb{R}$

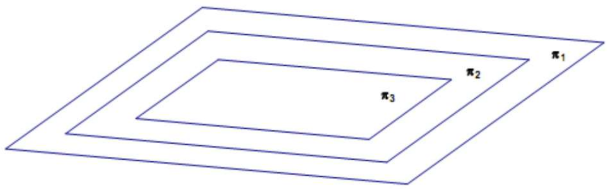
Steps: ① Analyze $\vec{n}_1, \vec{n}_2, \vec{n}_3$ ② Analyze D_1, D_2, D_3 $\xleftrightarrow{\text{or}}$ check $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$?

$\updownarrow$ or

Directly solve

Intersection of Three Planes

1. All three equations represent the same plane.



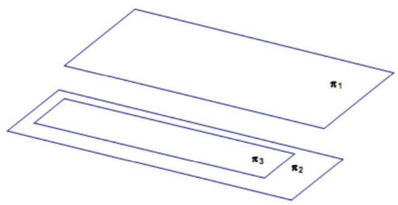
- normals are all scalar multiples of each other
- All three "D" values are the same scalar multiples.

$\vec{n}_1 = k\vec{n}_2 = s\vec{n}_2$

$D_1 = kD_2 = sD_3$, where $k, s \in \mathbb{R}$

- ➢ Parallel & Coincident
- ➢ Infinite solutions in the form of a plane equation

2. Two equations represent the same plane, the third is parallel.



$\vec{n}_1 = k\vec{n}_2 = s\vec{n}_3$

- normals are all scalar multiples of each other
- two "D" values are the same scalar multiples (not the third one).

$D_1 \neq kD_2 = sD_3$

- ➢ Two planes are Parallel & Coincident, one is Parallel & Distinct
- ➢ No solution

3. All three planes are parallel and distinct.

- normals are all scalar multiples of each other

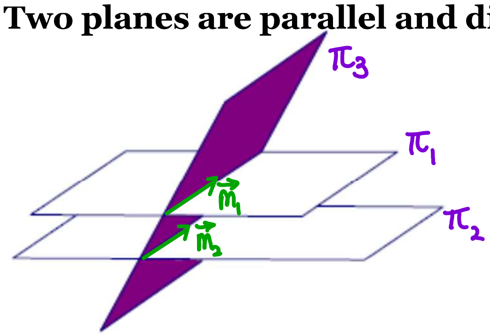
$\vec{n}_1 = k\vec{n}_2 = s\vec{n}_3$

- all "D" values are distinct

$D_1 \neq kD_2 \neq sD_3$

- ➢ All three planes are Parallel & Distinct
- ➢ No solution

4. Two planes are parallel and distinct, the third is not parallel.



- two planes have normals that are all scalar multiples of each other

$\vec{n}_1 = k\vec{n}_2 \neq s\vec{n}_3$

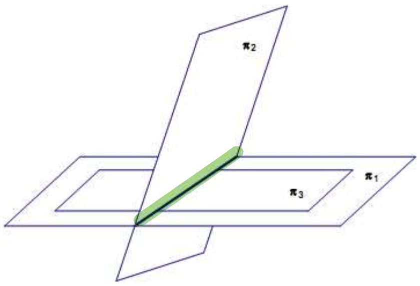
- direction vectors of the two lines of intersection are scalar multiples

$D_1 \neq kD_2$

$\vec{m}_1 = t\vec{m}_2$

- ➢ Two planes are Parallel & Distinct, one plane intersects both at parallel lines.(No solution)

5. Two of the equations represent the same plane, the third plane is non-parallel.



- two planes have normals and "D" values that are scalar multiples (coincident)
- the third plane intersects both planes

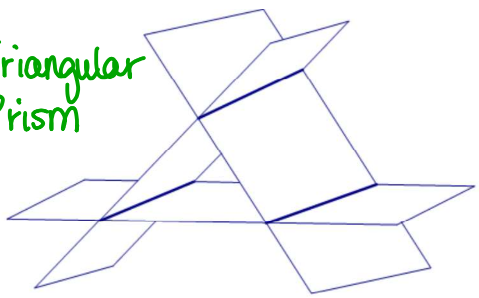
$\vec{n}_1 = s\vec{n}_3 \neq k\vec{n}_2$

$D_1 = sD_3$

- ➢ Two planes are Parallel & Coincident, one plane intersects both at a single line.
- ➢ Infinite solutions in the form of a line → find the line of intersection between π_1 and π_2 or π_2 and π_3

6. None of the three planes are parallel – no single intersection.

Triangular Prism



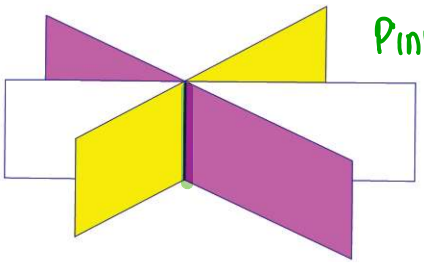
- The normals are distinct but not parallel (they are coplanar)

$\vec{n}_1 \neq k\vec{n}_2 \neq s\vec{n}_3$

Check triple scalar product to see if normals are coplanar. ie. $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$

- ➢ Pairs of planes intersect in 3 parallel lines
- ➢ No solution

7. None of the three planes are parallel – intersect in a line.



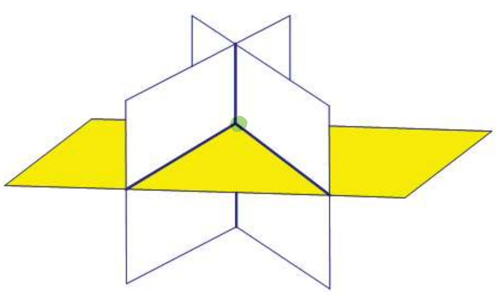
Pinwheel

- The normals are coplanar but not parallel

$\vec{n}_1 \neq k\vec{n}_2 \neq s\vec{n}_3$ but $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$

- ➢ All three planes intersect at one line
- ➢ Infinite solutions in the form of a line → solve for line of intersection between any 2 planes.

8. None of the three planes are parallel – intersect in a point.



- The normals are not parallel and not coplanar

$\vec{n}_1 \neq k\vec{n}_2 \neq s\vec{n}_3$

$\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$

- ➢ All three planes intersect at one point

One distinct solution in the form of a point

For cases #6,7,8 → solve using elimination to distinguish the cases. Solutions can be:
1) Contradiction (ie. 0=3) → Case #6 (Triangular Prism) – no solution
2) True Statement (ie. 0=0) → Case #7 (Pinwheel) – line of intersection
3) Unique values for variables → Case #8 – single point of intersection.

Intersection Of Three Planes

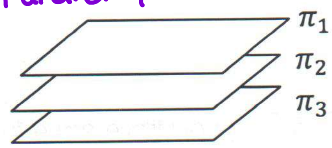
$Ax+By+Cz+D=0$

Major Step: Determine how many normal vectors are parallel to each other

Case 1: Three normals are parallel to each other. i.e. $\vec{n_1} = k\vec{n_2} = s\vec{n_3}, \quad k, s \epsilon R$

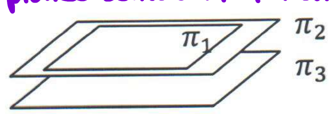
- Therefore, we have 3 parallel planes.

Parellel & distinct



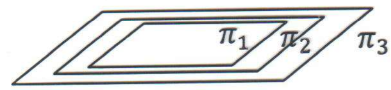
No pair of D values are scalar multiples

2 planes coincident & 1 distinct



One pair of D values are scalar multiples $D_1 = kD_2$

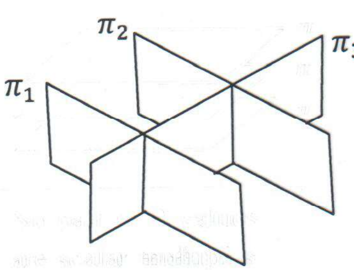
3 planes parallel and coincident

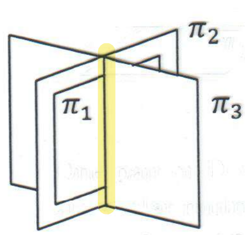


Two pairs of D values are scalar multiples $D_1 = kD_2 = sD_3$

Case 2: Two normals are parallel to each other.

- Therefore, we have 2 parallel planes





Check to see if two planes are coincident.

- **If not**, you have the first picture
  - **No Intersection**
- **If so**, you have the second picture
  - Planes **intersect in a line**

Note: For the second picture: $\vec{n_1} = k\vec{n_2}$ and $D_1 = kD_2$

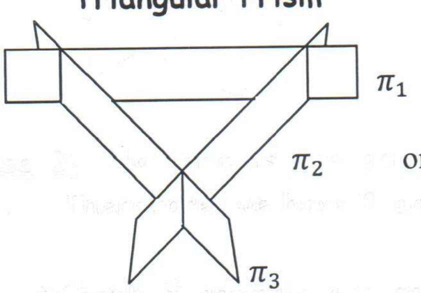
Case 3: No normals are parallel to each other

- Therefore, we have 0 parallel planes

→**Check if normals are coplanar** i.e. Perform $\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) = 0$??? or **Directly Solve**

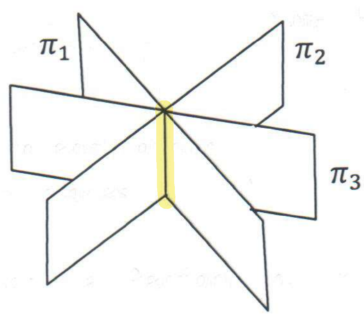
If normals are coplanar (i.e. $\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) = 0$) you have:

Triangular Prism



or

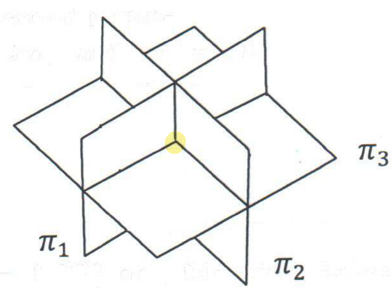
Pinwheel



No intersection

Line of intersection

If normals are not coplanar (i.e. $\vec{n_1} \cdot (\vec{n_2} \times \vec{n_3}) \neq 0$) you have:



One point of intersection

→You must solve the system to determine which case.

Strategy: Use ① and ② to get ④, then ① and ③ to get ⑤. Use ④ and ⑤ to solve for variables.

Find: 1) Contradiction (0=3) 2) True Statement (0=0) or 3) Unique values for variables

Definitions:
A system of three planes is **consistent** if it has one or more solutions.
A system of three planes is **inconsistent** if it has no solution.

Method for solving (possible) intersection of 3 planes

1. Check the normals for parallel or coincident planes. Suppose three distinct planes have normal vectors $\vec{n}_1, \vec{n}_2, \vec{n}_3$. To determine if there is a unique point of intersection, calculate $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3)$
   - If $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3) \neq 0$, the normal vectors are not coplanar.
     - **There is a single point of intersection**
   - If $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3) = 0$, the normal vectors are coplanar.
     - **There may or may not be points of intersection**
       - If there are any points of intersection then they lie on a line

2. Using Π_1 and Π_2, solve for 2 variables in terms of the third (similar to intersection of 2 planes).
3. Substitute new equations into Π_3 and interpret results:
   a) Point (solution of parameter eg. $t = 4$)
   b) Line (infinite possible # of solutions for parameter eg. $0t = 0$)
   c) Plane (infinite possible # of solutions eg. $\Pi_1 = k\Pi_2 = m\Pi_3$)
   d) None (if $0t = \#$)

Introduction to Matrices

Mathematicians often develop new notation and ideas to help ease complicated calculations or procedures. The process of solving the system can be simplified by using a matrix to help organize and eliminate variables efficiently.
Informally, a **matrix** (the plural form is matrices) is an array of m rows × n columns.
For example,

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -4 & 5 \\ 1 & 2 & 7 \end{bmatrix}$$

A is a 3×3 matrix since it has 3 rows and 3 columns.
We often use a single, capital letter to represent a matrix, such as A in our example.
Further, A_{ij} is the notation used to reference the element in the i^{th} row and j^{th} column of matrix A. In this example, $A_{31} = 1$.

ROW MATRIX: $\begin{bmatrix} 1 & 7 & 5 & -1 \end{bmatrix}$ ***COLUMN MATRIX:*** $\begin{bmatrix} 6 \\ 0 \\ 15 \end{bmatrix}$ ***SQUARE MATRIX:*** $\begin{bmatrix} 6 & -1 & 7 \\ 5 & 7 & 9 \\ -4 & -5 & 0 \end{bmatrix}$

To translate a system of linear equations into matrix form, we write the coefficients and the constant terms of the linear equations as elements in the corresponding locations in the matrix. From example

$$\begin{aligned} 1x + 2y - 4z &= 3 \\ -2x + 1y + 3z &= 4 \\ 4x - 3y - 1z &= -2 \end{aligned}$$

becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

Such a matrix that includes the constant terms is known as an **augmented** matrix, and the elements in the matrix to the left of the vertical line form the **coefficient** matrix.

Gaussian Elimination

The benefit of using matrices becomes obvious when solving linear systems using a process known as Gaussian elimination.
In Gaussian elimination, the aim is to transform a system of equations in augmented matrix form, such as

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

into another augmented matrix of the form

$$D = \left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

where D is known as a matrix in row **echelon form**.

Row Operations

To transform the original augmented matrix into row echelon form, we perform row operations. Possible row operations are as follows:

- Multiplying each entry in one row by a (non-zero) scalar
- Adding (or subtracting) one row to (from) another
- A combination of the above two row operations
- Interchanging rows

Examples: Investigate the intersection of the planes.

$\pi_1 : 2x - y + 3z - 2 = 0$
$\pi_2 : 4x - 2y + 6z - 3 = 0$
$\pi_3 : x - 3y + 2z + 10 = 0$

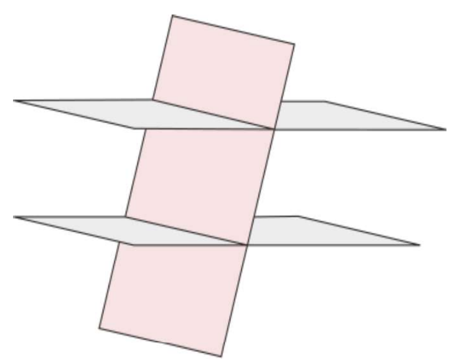
$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 4 & -2 & 6 & 3 \\ 1 & -3 & 2 & -10 \end{array}\right] R_1 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 4 & -2 & 6 & 3 \\ 2 & -1 & 3 & 2 \end{array}\right] (-2)\times R_3 + R_2 \rightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 4 & -2 & 6 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The third row of the matrix corresponds to the equation $0y = -1$, which has no solution. Thus, this system of equations has no solution and therefore, the three corresponding planes have no points of intersection.

To visualize this geometrically, we note that

$$\vec{n}_2 = 2\vec{n}_1$$
$$[4,-2,6] = 2[2,-1,3]$$
$$D_2 \neq 2D_1$$

The normal vector of the third plane, $\vec{n}_3 = [1,-3,2]$ is not parallel to either of these, so the third plane must intersect each of the other two planes in a line.



Using your GDC:

1. Press $\boxed{2^{nd}}$ $\boxed{x^{-1}}$ for the "matrix" menu. Cursor to the right to the EDIT menu. Choose the number for the matrix you want to edit. Enter the number of rows, and the number of columns you want for the matrix when prompted for $\boxed{} \times \boxed{}$, then press $\boxed{\text{enter}}$. For example 3x4 for a system of 3 equations.
2. Enter the values of the coefficients and the constant from the system into the matrix. When you are finished entering the values, press $\boxed{2^{nd}}$ $\boxed{\text{mode}}$ to go back to the home screen.
3. Press $\boxed{2^{nd}}$ $\boxed{x^{-1}}$ for the "matrix" menu, cursor to the right to the MATH menu. Scroll down and choose B: rref(for "reduced row echelon form" then press enter.
4. When prompted for the matrix name, press $\boxed{2^{nd}}$ $\boxed{x^{-1}}$ for the "matrix" menu, then choose the number for the matrix you are working with under the NAMES menu. Press enter.
5. The screen should give you the solution in the form of $\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ where the solution is $x = a, y = b, z = c$.

Investigate the intersection of the following planes.

1) $\pi_1 : 2x - y + 3z - 2 = 0$ — ①

$\pi_2 : x - 3y + 2z + 10 = 0$ — ②

$\pi_3 : 5x - 5y + 8z + 6 = 0$ — ③

Step ①: $\vec{n}_1 = [2,-1,3]$ $\quad \vec{n} = [1,-3,2]$ $\quad \vec{n}_3 = [5,-5,8]$

No pair is collinear

Check for coplanarity by evaluating $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)$:

$$[2,-1,3] \cdot ([1,-3,2] \times [5,-5,8])$$

$$\begin{array}{cccccc} 1 & -3 & 2 & 1 & -3 & 2 \\ 5 & -5 & 8 & 5 & -5 & 8 \end{array}$$

$$= [2,-1,3] \cdot [-24+10, 10-8, -5+15]$$

$$= [2,-1,3] \cdot [-14, 2, 10]$$

$$= -28 - 2 + 30$$

$$= 0$$

Since the triple scalar product is zero, the three normals are coplanar. Hence, the planes either have a line in common or no point in common.

Step ②: Solve the system:

①$-2\times$②:

$$\begin{aligned} 2x - y + 3z - 2 &= 0 \\ 2x - 6y + 4z + 20 &= 0 \\ \hline 5y - z &= 22 \quad \text{— ④} \end{aligned}$$

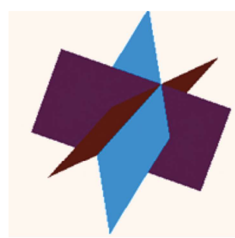
$5\times$②$-$③:

$$\begin{aligned} 5x - 15y + 10z + 50 &= 0 \\ 5x - 5y + 8z + 6 &= 0 \\ \hline -10y + 2z &= -44 \quad \text{— ⑤} \end{aligned}$$

$2\times$④$+$⑤:

$$\begin{aligned} 10y - 2z &= 44 \\ -10y + 2z &= -44 \\ \hline 0 &= 0 \end{aligned}$$

True statement $\Rightarrow$ infinite solutions



From ④: $z = 5y - 22$

From ②: $x = 3y - 2z - 10$

Let $y = t$

$$z = 5t - 22$$

$$x = 3t - 2(5t - 22) - 10$$

$$x = 3t - 10t + 44 - 10$$

$$x = -7t + 34$$

$\therefore$ Equation of the line of intersection is:

$$\vec{r} = [34, 0, -22] + t[-7, 1, 5], \; t \in \mathbb{R}$$

2) $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & -3 & 2 & -10 \\ 5 & -5 & 8 & -6 \end{array}\right]$

II

Π_1: $2x-y+3z-2=0$

Π_2: $x-3y+2z+10=0$

Π_3: $5x-5y+8z+6=0$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & -3 & 2 & -10 \\ 5 & -5 & 8 & -6 \end{array}\right]$$

$$R_2 \leftrightarrow R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 2 & -1 & 3 & 2 \\ 5 & -5 & 8 & -6 \end{array}\right]$$

$$\xrightarrow[-5\times R_1+R_3 \rightarrow R_3]{-2\times R_1+R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 0 & 5 & -1 & 22 \\ 0 & 10 & -2 & +44 \end{array}\right]$$

$$-2\times R_2+R_3 \rightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 0 & 5 & -1 & 22 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} 5y-z=22 \\ x-3y+2z=-10 \end{cases}$$

Let $y=t,$

$$\begin{cases} z=-22+5t \\ x-3t+2(-22+5t)=-10 \rightarrow x=34-7t \end{cases}$$

Equation of line of intersection

$$\boxed{\vec{r}=[34,0,-22]+t[7,1,5]}$$

$$\pi_1 : x+3y-z+9=0 \quad \text{— } ①$$

3) $$\pi_2 : x-y+z-11=0 \quad \text{— } ②$$

$$\pi_3 : x+2y+4z-5=0 \quad \text{— } ③$$

$\vec{n}=[1,3,-1]$ $\quad \vec{n_2}=[1,-1,1]$ $\quad \vec{n_3}=[1,2,4]$ $\quad \therefore$ No pair is collinear

$$\vec{n_1}\cdot\vec{n_2}\times\vec{n_3}$$
$$=[1,3,-1].([1,-1,1]\times[1,2,4])$$
$$=[1,3,-1].[-4-2,1-4,2+1]$$
$$=[1,3,-1].[-6,-3,3]$$
$$=-6-9-3$$
$$=-18\neq 0$$

$$\begin{array}{cccccc} 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 1 & 2 & 4 \end{array}$$

The normals are not coplanar.
There is a point of intersection.

Solving the system:

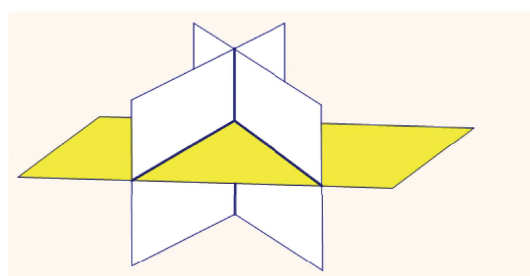
① − ② : $4y-2z=-20$ — ④

① − ③ : $y-5z=-14$ — ⑤

④ − 4×⑤ : $18z=36$

$$\boxed{z=2}$$

Sub in ⑤:
$$y=-14+5(2)$$
$$\boxed{y=-4}$$



Sub in ①:
$$x+3(-4)-2+9=0$$
$$\boxed{x=5}$$

$\therefore$ The 3 planes intersect at a point $(5,-4,2)$

$\pi_1 : 2x + 3y + z - 1 = 0$

4) $\pi_2 : x - y + z - 2 = 0$

$\pi_3 : 2x - 2y + 2z - 4 = 0$

$\vec{n_1} = [2,3,1] \quad D_1 = -1$

$\vec{n_2} = [1,-1,1] \quad D_2 = -2$

$\vec{n_3} = [2,-2,2] \quad D_3 = -4$

$= 2[1,-1,1] \quad = 2(-2)$

$\vec{n_3} = 2\vec{n_2} \quad D_3 = 2D_2$

$\therefore \pi_2 = \pi_3$ Planes are parallel and coincident

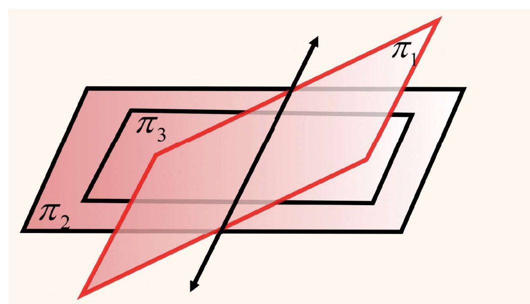
$\vec{m} = \vec{n_1} \times \vec{n_2}$

$= [2,3,1] \times [1,-1,1]$

$= [3+1, 1-2, -2-3]$

$= [4,-1,-5]$

$\left\{ \begin{array}{ccccc} 2 & 3 & 1 & 2 & 3 \\ 1 & -1 & 1 & 1 & -1 \end{array} \right\}$



Sub $y=0$ into $\pi_1 + \pi_2$:

$2x + z = 1$ —— ①

$x + z = 2$ —— ②

① − ②: $\boxed{x = -1}$

Sub in ②: $z = 2+1$

$\boxed{z = 3}$

$\therefore$ The equation of the line of intersection is :

$\vec{r} = [-1,0,3] + t[4,-1,-5], \; t \in \mathbb{R}$

5) $\begin{cases} 2x+y-3z+5=0 & —① \\ x+y+z-6=0 & —② \\ 4x-5y+z+3=0 & —③ \end{cases}$

$\vec{n}_1=[2,1,-3]$ $\quad \vec{n}_2=[1,1,1]$ $\quad \vec{n}_3=[4,-5,1]$ No pair is collinear.

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 4 & -5 & 1 & 4 & -5 \end{array}$$

$$\begin{aligned}
&\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 \\
&=[2,1,-3]\cdot\left([1,1,1]\times[4,-5,1]\right) \\
&=[2,1,-3]\cdot[1+5,4-1,-5-4] \\
&=[2,1,-3]\cdot[6,3,-9] \\
&=12+3+27 \\
&=42\neq 0
\end{aligned}$$

The normals are not coplanar. There is a point of intersection.

$$\begin{aligned}
①-2\times②:\ & 2x+y-3z+5=0 \\
& \underline{2x+2y+2z-12=0} \\
& -y-5z=-17 \\
& y=17-5z \quad —④
\end{aligned}$$

$$\begin{aligned}
4\times②-③:\ & 4x+4y+4z-24=0 \\
& \underline{4x-5y+z+3=0} \\
& 9y+3z=27 \\
& 3y+z=9 \quad —⑤
\end{aligned}$$

$$\begin{aligned}
\text{Sub ④ in ⑤}:\ & 3(17-5z)+z=9 \\
& 51-15z+z=9 \\
& -14z=-42 \\
& \boxed{z=3}
\end{aligned}$$

$$\begin{aligned}
\text{Sub in ⑤}:\ & 3y+3=9 \\
& 3y=6 \\
& \boxed{y=2}
\end{aligned}$$

$$\begin{aligned}
\text{Sub in ②}:\ & x+2+3-6=0 \\
& \boxed{x=1}
\end{aligned}$$

$\therefore$ The 3 planes intersect at a point $(1,2,3)$.

Practice

1. Given the related augmented matrix, find the solution to the system

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 0 & 2 & 0 & 6 \\ 0 & 0 & -3 & 9 \\ 4 & 0 & 0 & 8 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 12 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 7 \\ 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & -5 \end{array}\right]$

2. Use Gauss-Jordan elimination to find the intersection of each of the following sets of planes

a. $\begin{aligned} x+3y-z+9&=0 \\ x-y+z-11&=0 \\ x+2y+4z-5&=0 \end{aligned}$

b. $\begin{aligned} x-y-2z-5&=0 \\ 2x+2y+z-1&=0 \\ x+3y+3z-10&=0 \end{aligned}$

c. $\begin{aligned} 2x+3y+z-1&=0 \\ x-y+z-2&=0 \\ 2x-2y+2z-4&=0 \end{aligned}$

d. $\begin{aligned} 2x-y+3z&=0 \\ -2x-3y+2z+10&=0 \\ 5x-5y+8z+6&=0 \end{aligned}$

3. Find value(s) of a and b so that the planes

$2x+y-z=0$

$x+2y+3z=0$

$3x+ay+2z+b=0$

(a) intersect in a line
(b) intersect at a point.

4. For what value of k will the following set of planes interesting in a line?

$\pi_1 : x - 2y - 3z = 0$

$\pi_2 : x + 9y - 5z = 0$

$\pi_3 : kx - y + z = 0$

Answers

1. a. $(3,2,-5)$
   b. $(2,3,-3)$
   c. $(-4,-1,4)$
   d. No solution.

2. a. $(5,-4,2)$
   b. No solution.
   c. $\vec{r}=[0,2,0]+t[2,1,-1]$
   d. $\vec{r}=[\frac{16}{5},\frac{22}{5},0]+t[7,-1,5]$

3. a. $a=3, b=0$
   b. $a \neq 3, b \in R$

4. $k=\frac{-9}{37}$

Warm -up

Determine the intersection of the following planes

a. $\begin{aligned} x - y + z &= -2 \\ 2x - y - 2z &= -9 \\ 3x + y - z &= -2 \end{aligned}$ Ans: $[(-1,3, 2)]$

b. $\begin{aligned} x + y + 2z &= -2 \\ 3x - y + 14z &= 6 \\ x + 2y &= -2 \end{aligned}$ Ans: $[x, y, z] = [1, -3, 0] + [-4, 2, 1]$

c. $\begin{aligned} 3x + 2y - z &= 0 \\ 3x - 5y + 4z &= 3 \\ 2x - y + z &= 1 \end{aligned}$ Ans: $\vec{r} = [\frac{2}{7}, -\frac{3}{7}, 0] + t[-1, 5, 7]$

d. $\begin{aligned} x - y + 4z &= 5 \\ 3x + y + z &= -2 \\ 5x - y + 9z &= 1 \end{aligned}$ Ans: No intersection exists

1. Find the distance between the following skew lines

a) $L_1 : [x,y,z] = [-2,1,1] + t[3,1,1]; t \in R$
$L_2 : [x,y,z] = [7,-4,3] + s[-1,1,0]; s \in R$

b) $\ell_1 : \frac{x}{3} = \frac{y-4}{4} = z-8$
$\ell_2 : \frac{x+1}{2} = \frac{y-5}{5} = \frac{z+2}{2}$

2. **Solve** the following systems of equations. Give a **geometrical interpretation** of the system and its solution.

a)
$\pi_1 : x - 5y + 2z - 10 = 0$
$\pi_2 : x + 7y - 2z + 6 = 0$
$\pi_3 : 8x + 5y + z - 20 = 0$

b)
$\pi_1 : 2x + y + 6z - 7 = 0$
$\pi_2 : 3x + 4y + 3z + 8 = 0$
$\pi_3 : x - 2y - 4z - 9 = 0$

3. Determine the shortest **exact** distance from the point $A(1, 2, 3)$ and the line $\begin{cases} x = 3s \\ y = 1 + 4s \\ z = 5 + s \end{cases}$.

4. The shortest distance between the point (1,3,2) and the line $\frac{x-1}{-1} = \frac{y-3}{1} = \frac{z-k}{2}$ is $\sqrt{3}$ units. Find the value(s) of k.

5. Consider the following three planes:
   ① $4x + 3y + 3z = -8$
   ② $2x + y + z = -4$
   ③ $3x - 2y + (m^2 - 6)z = m - 4$

   Determine the value(s) of m for which the planes intersect in a point.

6. Prove that the following planes may intersect in a line provided that $a^2 + b^2 + c^2 = 1 - 2abc$.
   $\pi_1 : -x + ay + bz = 0$
   $\pi_2 : ax - y + cz = 0$
   $\pi_3 : bx + cy - z = 0$

1. Find the distance between the following skew lines

a) $L_1 : [x,y,z]=[-2,1,1]+t[3,1,1]; t\in R$
$L_2 : [x,y,z]=[7,-4,3]+s[-1,1,0]; s\in R$

$P=(-2,1,1)$ & $Q(7,-4,3)$

$\overrightarrow{PQ}=[9,-5,2]$

$\overrightarrow{m_1}\times\overrightarrow{m_2}=[3,1,1]\times[-1,1,0]$
$=[-1,-1,4]$

$\overrightarrow{PQ}\bullet(\overrightarrow{m_1}\times\overrightarrow{m_2})=[9,-5,2]\bullet[-1,-1,4]$
$=4$

$|\overrightarrow{m_1}\times\overrightarrow{m_2}|=\sqrt{1^2+1^2+4^2}=3\sqrt{2}$

$$d=\frac{|\overrightarrow{PQ}\bullet(\overrightarrow{m_1}\times\overrightarrow{m_2})|}{|\overrightarrow{m_1}\times\overrightarrow{m_2}|}=\frac{|4|}{3\sqrt{2}}=\frac{2\sqrt{2}}{3}\text{ units}$$

b) $\ell_1 : \frac{x}{3}=\frac{y-4}{4}=z-8$

$\ell_2 : \frac{x+1}{2}=\frac{y-5}{5}=\frac{z+2}{2}$

$P=(0,4,8)$ & $Q(-1,5,-2)$

$\overrightarrow{PQ}=[-1,1,-10]$

$\overrightarrow{m_1}\times\overrightarrow{m_2}=[3,4,1]\times[2,5,2]$
$=[3,-4,7]$

$\overrightarrow{PQ}\bullet(\overrightarrow{m_1}\times\overrightarrow{m_2})=[-1,1,-10]\bullet[3,-4,7]$
$=-77$

$|\overrightarrow{m_1}\times\overrightarrow{m_2}|=\sqrt{3^2+4^2+7^2}=\sqrt{74}$

$$d=\frac{|\overrightarrow{PQ}\bullet(\overrightarrow{m_1}\times\overrightarrow{m_2})|}{|\overrightarrow{m_1}\times\overrightarrow{m_2}|}=\frac{|-77|}{\sqrt{74}}=8.95\text{ units}$$

2. **Solve** the following systems of equations. Give a **geometrical interpretation** of the system and its solution.

a)
$\pi_1 : x-5y+2z-10=0$
$\pi_2 : x+7y-2z+6=0$
$\pi_3 : 8x+5y+z-20=0$

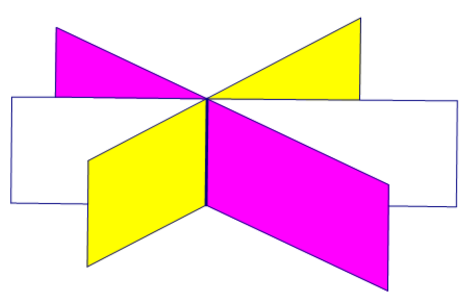
$$\left[\begin{array}{ccc|c}1&-5&2&10\\1&7&-2&-6\\8&5&1&20\end{array}\right]\xrightarrow[-8R_1+R_3\to R_3]{-R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1&-5&2&10\\0&12&-4&-16\\0&45&-15&-60\end{array}\right]\xrightarrow[\frac{1}{15}R_3\to R_3]{\frac{1}{4}R_2\to R_2}\left[\begin{array}{ccc|c}1&-5&2&10\\0&3&-1&-4\\0&3&-1&-4\end{array}\right]$$

$$\xrightarrow{-R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1&-5&2&10\\0&3&-1&-4\\0&0&0&0\end{array}\right]$$

$3y-z=-4$ and $x-5y+2z=10$

Let $y=t$ then $z=3t+4$ and $x=5t-2(3t+4)+10$ or $x=-t+2$

Equation of line of intersection: $\vec{r}=[2,0,4]+t[-1,1,3]$ $,t\in R$



b) $\pi_1 : 2x + y + 6z - 7 = 0$
$\pi_2 : 3x + 4y + 3z + 8 = 0$
$\pi_3 : x - 2y - 4z - 9 = 0$

$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 7 \\ 3 & 4 & 3 & -8 \\ 1 & -2 & -4 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 9 \\ 3 & 4 & 3 & -8 \\ 2 & 1 & 6 & 7 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 9 \\ 0 & 10 & 15 & -35 \\ 0 & 5 & 14 & -11 \end{array}\right]$$

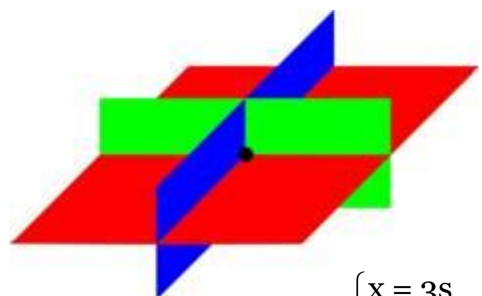
$$\xrightarrow{\frac{1}{10}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 9 \\ 0 & 1 & 1.5 & -3.5 \\ 0 & 5 & 14 & -11 \end{array}\right] \xrightarrow{-5R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -2 & -4 & 9 \\ 0 & 1 & 1.5 & -3.5 \\ 0 & 0 & 6.5 & 6.5 \end{array}\right]$$

$6.5z = 6.5 \Rightarrow \boxed{z = 1}$

$y + 1.5z = -3.5 \xrightarrow{z=1} \boxed{y = -5}$

$x - 2y - 4z = 9 \xrightarrow{z=1 \ \& \ y=-5} \boxed{x = 3}$

Point of intersection: (3,-5,1)



3. Determine the shortest **exact** distance from the point $A(1, 2, 3)$ and the line $\begin{cases} x = 3s \\ y = 1 + 4s \\ z = 5 + s \end{cases}$.

$A(1,2,3)$, $B(0,1,5)$ *[B lies on the line]*

$\overrightarrow{AB} = [-1,-1,2]$, $\vec{m} = [3,4,1]$

$$\overrightarrow{AB} \times \vec{m} = [-1,-1,2] \times [3,4,1] = [-9,7,-1]$$

$$d = \frac{\left|\overrightarrow{AB} \times \vec{m}\right|}{\left|\vec{m}\right|} = \frac{\sqrt{9^2 + 7^2 + 1^1}}{\sqrt{3^2 + 4^2 + 1^1}} = \frac{\sqrt{131}}{\sqrt{26}} \text{ or } \frac{\sqrt{3406}}{26} \text{ units}$$

4. The shortest distance between the point (1,3,2) and the line $\frac{x-1}{-1}=\frac{y-3}{1}=\frac{z-k}{2}$ is $\sqrt{3}$ units. Find the value(s) of k .

P(1,3,2) and Q(1,3,k)

$\overrightarrow{PQ}=[1,3,k]-[1,3,2]=[0,0,k-2]$; $\vec{m}=[-1,1,2]$

$\overrightarrow{PQ}\times\vec{m}=[-1,-1,2]\times[0,0,k-2]$

$=[2-k,2-k,0]$

$|\overrightarrow{PQ}\times\vec{m}|=\sqrt{2(2-k)^2}$

$|\vec{m}|=\sqrt{1+1+4}$

$=\sqrt{6}$

$d=\frac{|\overrightarrow{P_oQ}\times\vec{m}|}{|\vec{m}|}$

$\sqrt{3}=\frac{\sqrt{2(2-k)^2}}{\sqrt{6}}$

$\sqrt{18}=\sqrt{2(2-k)^2}$

$18=2(2-k)^2$

$9=(2-k)^2$

$\pm3=2-k$

$3=2-k$ or $-3=2-k$

$\boxed{k=-1}$ $\boxed{k=5}$

5. Consider the following three planes:

① $4x+3y+3z=-8$

② $2x+y+z=-4$

③ $3x-2y+(m^2-6)z=m-4$

Determine the value(s) of m for which the planes intersect in a point.

$$\left[\begin{array}{ccc|c} 4 & 3 & 3 & -8 \\ 2 & 1 & 1 & -4 \\ 3 & -2 & m^2-4 & m-4 \end{array}\right] \xrightarrow{R_2\leftrightarrow R_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 4 & 3 & 3 & -8 \\ 3 & -2 & m^2-4 & m-4 \end{array}\right] \xrightarrow[-\frac{3}{2}R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{-7}{2} & m^2-\frac{15}{2} & 2m+4 \end{array}\right]$$

$$\xrightarrow{\frac{7}{2}R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 2 & 1 & 1 & -4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2m^2-8 & 2m+4 \end{array}\right]$$

To have only one point of intersection we must have $m\neq-2$.

6. Prove that the following planes may intersect in a line provided that $a^2 + b^2 + c^2 = 1 - 2abc$.

$$\pi_1 : -x + ay + bz = 0$$
$$\pi_2 : ax - y + cz = 0$$
$$\pi_3 : bx + cy - z = 0$$

Three planes intersect in a line iff $\vec{n}_1 \bullet \left(\vec{n}_2 \times \vec{n}_3 \right) = 0.$

$$[-1, a, b] \bullet ([a, -1, c] \times [b, c, -1]) = 0$$
$$[-1, a, b] \bullet [1 - c^2, cb + a, ac + b] = 0$$
$$-1 + c^2 + a(cb + a) + b(ac + b) = 0$$
$$-1 + c^2 + acb + a^2 + bac + b^2 = 0$$
$$\therefore a^2 + b^2 + c^2 = 1 - 2abc$$

Quiz: Equations Of Planes Name: _______________

1. Determine two points on the plane $\overrightarrow{OP} = [2,0,-1] + s[0,2,-1] + t[5,3,4]$.

Sub $s=0, t=0 \Rightarrow$ point $(2,0,-1)$

Sub $s=1, t=0 \Rightarrow$ point $(2,2,-2)$

2. Determine the parametric equations for the plane $[x-2,y,z-3] = s[1,1,1]+t[0,3,5]$.

$\vec{r} = [2,0,3] + s[1,1,1] + t[0,3,5]$

Parametric equations $\Rightarrow$ $x = 2+s$

$y = s+3t$ $\quad s,t \in \mathbb{R}$

$z = 3+s+5t$

3. Give a vector equation for the plane described by $x=3-s+4t$, $y=2t$, $z=1+4s-5t$.

$\vec{r} = [3,0,1] + s[-1,0,4] + t[4,2,-5]$ $\quad s,t \in \mathbb{R}$

4. Find a normal to the plane $\vec{r} = [3,1,-6] + s[0,1,1] + t[-1,2,1]$.

$\vec{n} = [0,1,1] \times [-1,2,1]$

$\vec{n} = [1-2, -1-0, 0+1]$

$\vec{n} = [-1,-1,1]$

$\begin{matrix} 0 & 1 & 1 & 0 & 1 \\ -1 & 2 & 1 & -1 & 2 \end{matrix}$

5. Find vector and parametric equations of each plane.

a) plane parallel to xy- plane through the point (l, 3, 5).

$\vec{r} = [1,3,5] + s[1,0,0] + t[0,1,0]$ $\quad s,t \in \mathbb{R}$

Note Any direction vector can be used as long as there is no z-component

$x = 1+s$

$y = 3+t$ $\quad s,t \in \mathbb{R}$

$z = 5$

b) The plane through (3, -5, 1) and parallel to the plane given by

$$x = s+3t-5, y = 2+s-t, z = s+t-5$$

or $\vec{r} = [-5,2,-5] + s[1,1,1] + t[3,-1,1]$, $s,t \in \mathbb{R}$

Vector equation of plane is

$\vec{r} = [3,-5,1] + m[1,1,1] + n[3,-1,1]$ $\quad m,n \in \mathbb{R}$

Parametric equations $\Rightarrow$ $x = 3+m+3n$

$y = -5+m-n$ $\quad m,n \in \mathbb{R}$

$z = 1+m+n$

c) The plane through the points $\overset{A}{(1, -3, 2)}$, $\overset{B}{(0, 1, -2)}$ and $\overset{C}{(9, 2, 0)}$.

$\vec{d_1} = \vec{AB} = [-1, 4, -4]$

$\vec{d_2} = \vec{AC} = [8, 5, -2]$

Vector equation of the plane is $\vec{r} = [1,-3,2] + s[-1,4,-4] + t[8,5,-2]$ $s,t \in \mathbb{R}$

Parametric equations are
$$x = 1 - s + 8t$$
$$y = -3 + 4s + 5t \qquad s,t \in \mathbb{R}$$
$$z = 2 - 4s - 2t$$

6. Find a scalar equation for each plane .

a) The plane with direction vectors [1, 5, -2] ,[0,0,1] through the point (3,4,1).

$\vec{n} = [1,5,-2] \times [0,0,1]$

$\vec{n} = [5,-1,0]$

$$\begin{array}{cccccc} 1 & 5 & -2 & 1 & 5 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

$5x - y + D = 0$

Sub $(3,4,1)$ $\quad 5(3) - 4 + D = 0$

$D = -11$

Note normal has no z-component, plane is parallel to z-axis (no z-intercept)

$\therefore$ The scalar equation is $5x - y - 11 = 0$

b) The plane containing the line $[x,y,z] = [1,3,-1] + r[3,2,2]$ and parallel to the

$x = 2 + 3s, y = -2s, z = 4 \Rightarrow \vec{r} = [2,0,4] + s[3,-2,0]$, $s \in \mathbb{R}$

$\vec{n} = \vec{d_1} \times \vec{d_2}$

$= [3,2,2] \times [3,-2,0]$

$= [0+4, 6-0, -6-6]$

$= [4,6,-12]$

$= 2[2,3,-6]$

$$\begin{array}{cccccc} 3 & 2 & 2 & 3 & 2 & 2 \\ 3 & -2 & 0 & 3 & -2 & 0 \end{array}$$

$\therefore 2x + 3y - 6z + D = 0$

Sub $(1,3,-1)$ $\quad 2(1) + 3(3) - 6(-1) + D = 0$

$2 + 9 + 6 + D = 0$

$D = -17$

The equation is $2x + 3y - 6z - 17 = 0$

c) The plane through the point (0,2,-1) and perpendicular to the line [x, y, z] =t[3,-2,4]

$\vec{d} = \vec{n} = [3,-2,4]$

$3x - 2y + 4z + D = 0$

Sub $(0,2,-1)$

$3(0) - 2(2) + 4(-1) + D = 0$

$-8 + D = 0$

$D = 8$

The scalar equation is $3x - 2y + 4z + 8 = 0$