Unit 5
Equations of Planes

Vector & Parametric Equations of a Plane

A Plane is:

- a flat surface that extends infinitely far in all directions
- represented by a parallelogram
- denoted by π

How can a plane be formed?

1. ______________________________
2. ______________________________
3. ______________________________
4. ______________________________

Developing the vector and parametric equations of a plane

Consider: How can we specify an arbitrary point, $P\ (x, y, z)$ on a plane, π ?

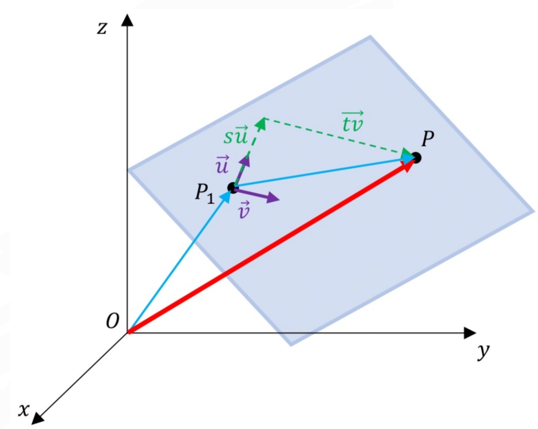
Recall: Three non-collinear points A,B, and P_1 determine a plane π .

by triangle law of addition :

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P}$$

$$\overrightarrow{P_1P} = s\vec{u} + t\vec{v}$$

$$\therefore \overrightarrow{OP} = \overrightarrow{OP_1} + s\vec{u} + t\vec{v}$$



Plane in R^3

Determined by a point P_1 (   ) and **two non collinear direction vectors**

$\vec{u}$ = [   ] and $\vec{v}$ = [   ]

Vector Equation

[,   ,   ]= [,   ,   ] + s [,   ,   ] + t [,   ,   ]

Parametric Equations of a Plane in R^3

$x =$

$y =$

$z =$

Example 1: The plane $\vec{r} = [1,0,0] + t[1,1,1] + s[1,-1,1]; s,t \in R$ contains the point $(3,0,k)$. Find the value of k .

Example 2: Find the vector and parametric equations of the plane that contains the points: A (1, 0, -3), B (2, -3, 1) and C (3, 5, -3).

Example 3: What does the following equation represent? Explain.

$$\vec{r} = [-1,2,3] + s[-7,21,14] + t[2,-6,-4]; t,s \in R$$

Example 4: Determine the vector equation of the plane containing the line $\begin{cases} x = 3 + 5t \\ y = -2 - 2t \\ z = 1 + 3t \end{cases}$ and the point P(1, 2, 3).

Practice

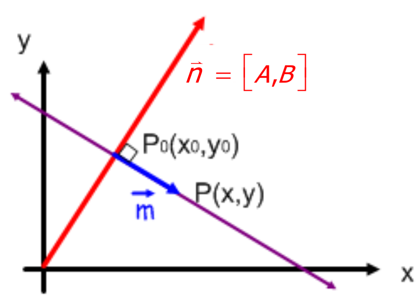
1. Find the **vector equation** of the plane that contains line $\frac{3-x}{2} = \frac{y+5}{-2} = \frac{z-4}{3}$ and the y-intercept of $[x,y,z]=[2,1,-4]+t[2,-1,-4], t \in R$.

2. Given $L_1 : \vec{r_1} = [3,2,-1] + t[1,2,-1]$ and $L_2 : \vec{r_2} = [1,-4,2] + s[-2,-4,2]$
   (a)Show that the lines are parallel and distinct.
   (b)Find a vector equation of the plane containing both lines

3. Find a vector equation of the plane that passes through the point (6, 0, 0) and contains the line $x = 4 - 2t$, $y = 2 + 3t$, $z = 3 + t$.

The Cartesian Equation of a Plane

Recall: For a line:



$\vec{m} = \overrightarrow{P_oP} = [x - x_o, y - y_o]$

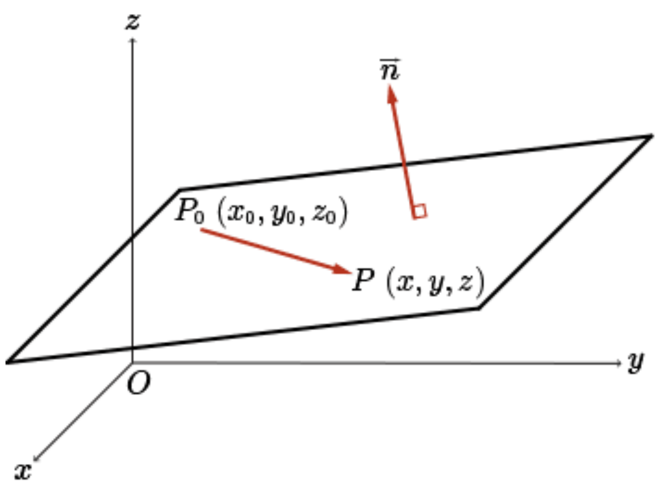
$\vec{n} = [A, B]$

$\therefore \vec{n} \bullet \vec{m} = 0$ (perpendicular)

$[A, B] \bullet [x - x_o, y - y_o] = 0$

$Ax + By + C = 0$ (Cartesian Equation of a Line)

For a plane:



If $P_0(x_0, y_0, z_0)$, $P(x, y, z)$ and $\vec{n} = [A, B, C]$ then $\overrightarrow{P_oP} \bullet \vec{n} = 0$,therefore:

Scalar Equation of a Plane in R^3

Ex. 1: Find the equation of π parallel to $2x - 3y + 4z - 3 = 0$ that contains the point A (1, 3, 5).

Ex.2: A line has vector equation $\vec{r} = [0,-5,2] + s[1,1,-2]$; $s \in R$ and lies on the plane π.The point P(2,-3,0) also lies on the plane π. Determine the Cartesian equation of this plane.

Ex.3: Determine the scalar equation of the plane that passes through the points P(6, −1, −1) and Q(3, 2, 1) and is perpendicular to the plane $x - 4y + 5z + 5 = 0$.

Practice

1. Determine the Cartesian equation of the plane that passes through the point A(6,-1,1), has a z-intercept of -4, and is parallel to the line $\vec{r}=[-2,-1,0]+t[3,3,-1]$.

2. Find scalar equation of a plane contains the line $\begin{cases} x = 1+t, \\ y = 1-t, \\ z = 2t, t \in R \end{cases}$ and is perpendicular to the plane with equation $x + y + z = 2$.

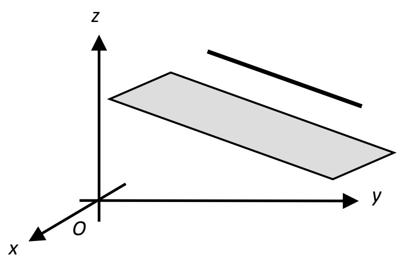
3. Find a Cartesian equation of the plane which passes through A(0, 1, -3), and is parallel to the lines

$$L_1 : \frac{x-1}{5} = \frac{y+2}{-2} = \frac{z+273}{-1} \qquad \text{and} \qquad L_1 : \frac{x+3}{-7} = \frac{y-1}{2}, z = 1$$

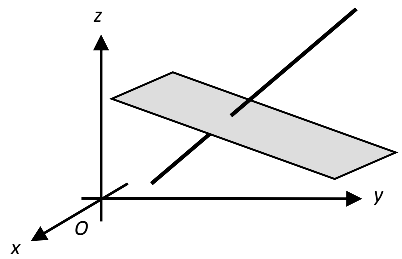
4. Find possible values of k such that the line [x, y, z] = [3, 4, 7] + t[k, 1, -2] is parallel to the plane 3kx + ky + z - 6 = 0.

Intersection of a Line and a Plane

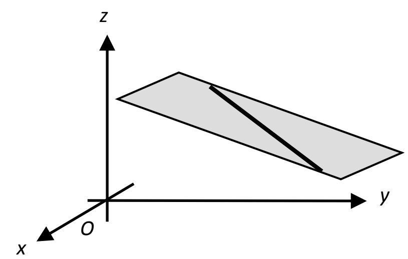
There are three ways for a line to interact with a plane in 3-space.



Line parallel is to the plane



Line intersects the plane



Line lies on the plane

Examples: Determine whether the line and plane intersect. If they do, find the point of intersection.

1) $L: \begin{cases} x = 1 + 2t \\ y = -6 + 3t \\ z = -5 + 2t \end{cases}$ $\qquad \boldsymbol{\pi}: \ 4x - 2y + z - 19 = 0$

2) $L: \vec{r} = [0,1,-4] + t[2,-1,1]$ $\qquad$ $\pi: \; x + 4y + 2z - 4 = 0$

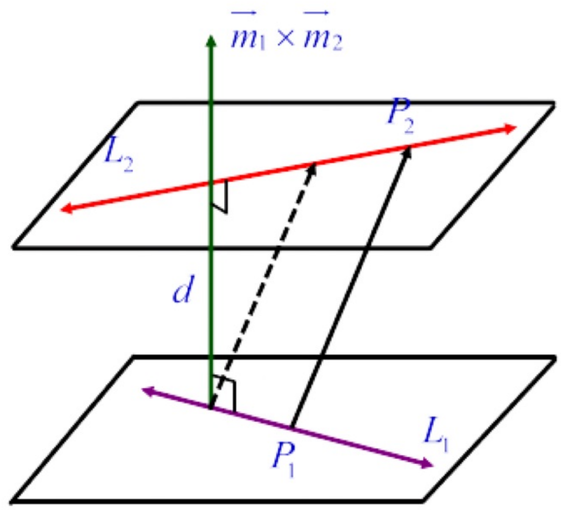
3) $L: \begin{cases} x = -4 + 3t \\ y = 0 \\ z = t \end{cases}$ $\qquad$ $\pi: \; x - 2y - 3z + 4 = 0$

4) Where does the line $\vec{r} = [6,10,-1] + t[3,4,-1]$ meet the xz-plane?

Distance between Skew Lines in R3

Let P_1 be a point on L_1 with direction vector $\vec{m}_1$ and let P_2 be a point on L_2 with direction vector $\vec{m}_2$ such that L_1 and L_2 are skew lines. The shortest distance between these skew lines is $d = \frac{\left|\overrightarrow{P_1P_2} \bullet (\vec{m}_1 \times \vec{m}_2)\right|}{\left|\vec{m}_1 \times \vec{m}_2\right|}$

Proof:



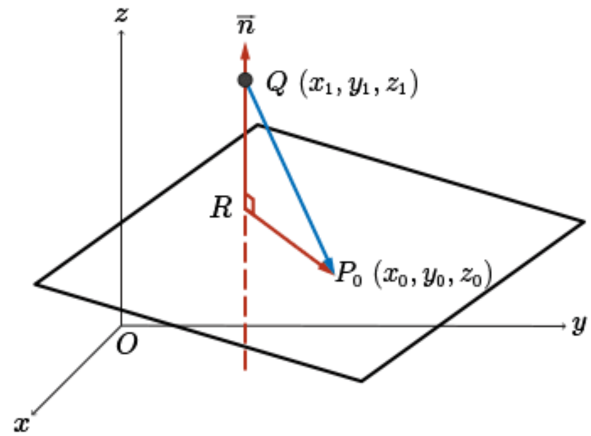
Example : Find the distance between the following skew lines
$[x, y, z] = [-1, 2, -8] + t[-3, 1, 2]$
$[x, y, z] = [2, 5, 11] + s[1, 2, 3]$.

The Distance between a Point and a Plane in R^3

Distance from a Point to a Plane

The distance from a point $Q(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is $d = \dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Proof:



Example: Find the shortest distance between the plane 4x + 2y + z – 16 = 0 and each point.

a) P(10,3,-8)

b) B(2,2,4)

Distance between Two Planes

Distance between two parallel planes with equations $Ax + By + Cz + D_1 = 0$, $Ax + By + Cz + D_2 = 0$ is: $d = \dfrac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$

Example: Find the shortest distance between the planes with equations 3x – 4y + 5z -10 = 0 and 6x – 8y + 10z – 10 = 0.

Exit Card!

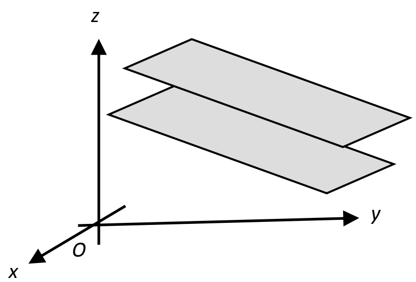
1) Determine the value(s) of ***k*** for which the planes $x - 2y + 2z + k = 0$ and $x - 2y + 2z = k - 2$ have a distance of 8 units.

2) The shortest distance between the skew lines $\begin{matrix}[x, y, z] = [1, 1, K] + t[2, -1, 1]; t \in R \\ [x, y, z] = [0, 0, 2] + s[3, 1, 2]; s \in R\end{matrix}$ is $\sqrt{35}$ units.

Find the value(s) of $\boldsymbol{K}$.

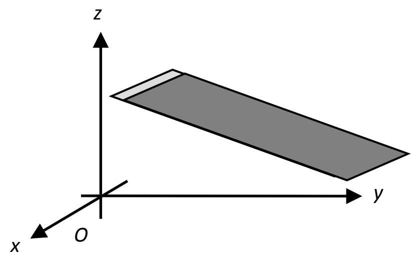
Intersection of Two Planes

There are three ways for 2 planes to interact in 3-space.



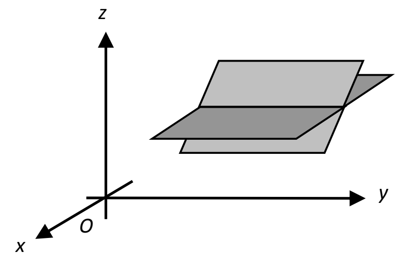
Parallel and distinct

(no intersection)



Parallel and coincident

(intersect at every point)



Intersect in a line

- The planes are parallel and distinct or coincident **iff** (if and only if) the normals are collinear (scalar multiples).
- The planes intersect **iff** the normals are not collinear.

Examples: Investigate the intersection of the planes and find the equation of the line of intersection if applicable.

1) $\pi_1 : 4x - 5y - 2z - 1 = 0$
$\pi_2 : x - y - 2z = 0$

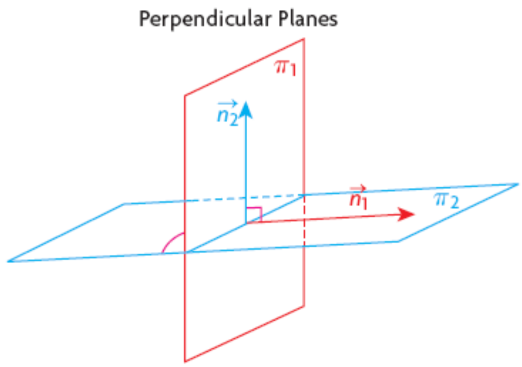
2) $\pi_1 : x + 2y + 3z - 6 = 0$

$\pi_2 : 4x + 8y + 12z = 25$

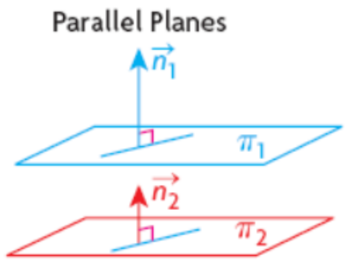
3) $\pi_1: \begin{cases} x = -4 + s \\ y = s + 3t \\ z = -s - 2t \end{cases}$ $\pi_2: \begin{cases} x = 8 + 6u + 2v \\ y = 3u + 5v \\ z = u - v \end{cases}$

Special Cases

Two planes are perpendicular iff __________

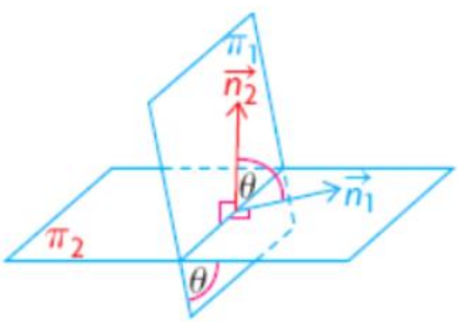


Two planes are parallel iff ___________________



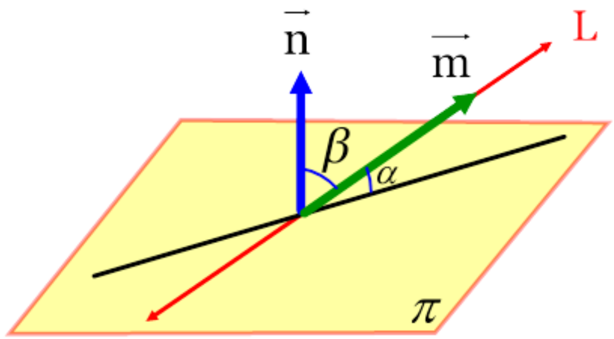
Angle between two planes:

The angle between two planes is the same as angle between their normals!



$$\cos\theta = \frac{\vec{n}_1 \bullet \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

Angle between a line and a plane



$$\cos\beta = \frac{\vec{n} \bullet \vec{m}}{|\vec{n}||\vec{m}|}$$

Ex. Line ℓ with equation $\vec{r} = [5,-1,4] + t[2,-2,0]$, $t \in R$ intersects the plane $\pi: \mathbf{a}x + z = 5\mathbf{a} + 4$ at an angle of $\frac{\pi}{6}$.Find the value of **a**, where **a** is a positive constant

Practice

1. Determine the intersection of the planes π_1: $6x - 3y + 12z = 9$ and π_2: $4x - 2y + 8z = 6$ if it exists. If it does, determine the acute angle between the two planes.

2. Determine the intersection of the planes $\pi 1$: $x + 3y + z = 2$ and $\pi 2$: $2x + y - z = -1$ if it exists. If it does, determine the acute angle between the two planes.

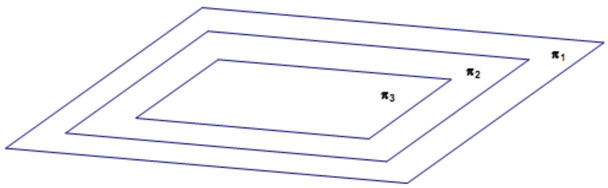
3. Find the intersection of $\pi 1$: $2x + y + 3z = -7$ and π_2 :$x - y + z = -5$ by using the cross product of the two normal vectors to find a direction vector of the line. When finding your point, set x=0 in both equations and solve for y and z.

4. The planes with equations π_1: $3x - 5y + 2z = 1$ and π_2: $3x - 5y + 2z = k$ are parallel $k \epsilon \boldsymbol{R}$.
a. A line intersects π_1 perpendicularly at the point (2, 1, 0), where does it intersect π_2?
b. Find all values of k for which the distance between the planes equals 3.

1. Given the line $\vec{r} = [12,-8,-4] + t[-3,4,2]$, find the intersection(s) with the xy- plane.
2. Find vector and parametric equations of the plane that contains the two intersecting lines , $\vec{r} = [3,-1,2] + s[4,0,1]$ and , $\vec{r} = [3,-1,2] + t[4,0,2]$.
3. Find a scalar equation of the plane that contains the origin and the point (2, -3, 2) and is perpendicular to the plane $x + 2y - z + 3 = 0$.
4. Determine the parametric equations for the plane containing the 2 parallel lines: $L_1 : \vec{r} = [0,1,3] + t[-6,-3,6]$ and $L_2 : \vec{r} = [-4,5,-4] + s[4,2,-4]$.
5. Determine the **scalar** equation of the plane parallel to the plane $\pi_1 : 3x + y - 2z - 4 = 0$ and containing the point of intersection of lines $L_1 : x = 7 + 2s, y = 2 + s, z = -6 - 3s$ and $L_2 : \vec{r} = [3,9,13] + t[1,5,5]$.
6. Find the scalar equation of the plane that is perpendicular to the plane $\vec{r} = [4,-5,2] + s[2,1,3] + t[-1,4,0]$ and intersects it at the line $\vec{r} = [4,-5,2] + t[1,-1,1]$.
7. Find the **vector** equation of the line of intersection between the plane $5x - 2y + 3z - 2 = 0$ and the *xy*-plane.
8. Consider the lines $l_1 : \vec{r} = [1,-2,4] + s[1,1,-3]$ and $l_2 : \vec{r} = [4,-2,k] + t[2,3,1]$ Determine the equation of the plane that contains l_1 and is parallel to l_2 .
9. Find the equation of the plane that contains the line $\vec{r} = [3,1,0] + t[2,1,4]$ and is perpendicular to the plane $\pi : \vec{p} = [1,1,1] + k[1,0,5] + s[-4,2,3]$.
10. Determine the measure of the acute angle to the nearest degree between the lines: $l_1 : \vec{r} = [1,-2,3] + t[4,1,-1]$ and $l_2 : \; x = 3 - t \;,\; y = t \;, z = 3 + 2t$.
11. Determine the Cartesian equation of the plane that contains the point (2,-1,1) and is perpendicular to the line joining points (-1,3,2) and (4,0,-2).
12. Determine the value of k if the acute angle between the planes 2x+ky-8=0 and x-3y+z+4=0 is 60°.

Intersection of Three Planes

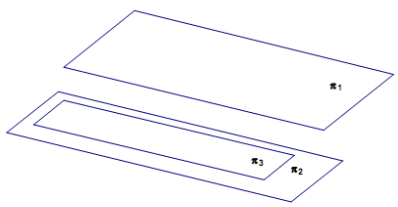
1. All three equations represent the same plane.



- normals are all scalar multiples of each other
- All three "D" values are the same scalar multiples.

➢ Parallel & Coincident
➢ Infinite solutions in the form of a plane equation

2. Two equations represent the same plane, the third is parallel.



- normals are all scalar multiples of each other
- two "D" values are the same scalar multiples (not the third one).

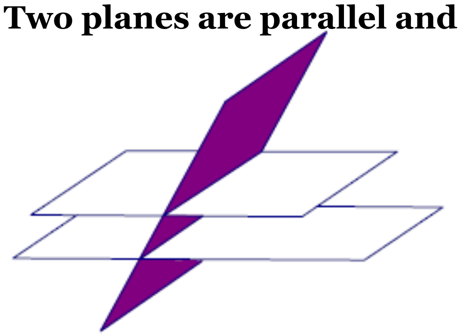
➢ Two planes are Parallel & Coincident, one is Parallel & Distinct
➢ No solution

3. All three planes are parallel and distinct.

- normals are all scalar multiples of each other
- all "D" values are distinct

➢ All three planes are Parallel & Distinct
➢ No solution

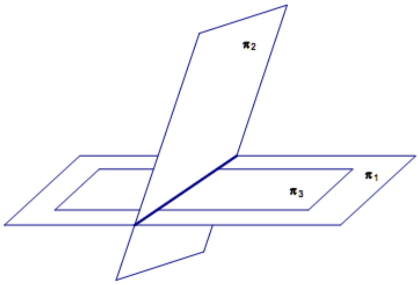
4. Two planes are parallel and distinct, the third is not parallel.



- two planes have normals that are all scalar multiples of each other
- direction vectors of the two lines of intersection are scalar multiples

➢ Two planes are Parallel & Distinct, one plane intersects both at parallel lines.(No solution)

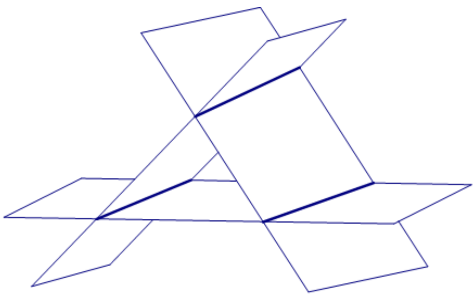
5. Two of the equations represent the same plane, the third plane is non-parallel.



- two planes have normals and "D" values that are scalar multiples (coincident)
- the third plane intersects both planes

➢ Two planes are Parallel & Coincident, one plane intersects both at a single line.
➢ Infinite solutions in the form of a line

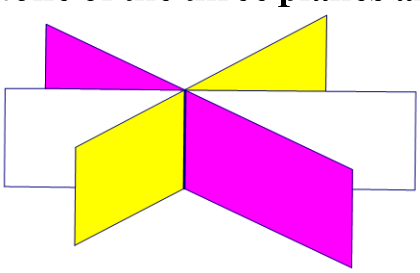
6. None of the three planes are parallel – no single intersection.



- The normals are distinct but not parallel

➢ Pairs of planes intersect in 3 parallel lines
➢ No solution

7. None of the three planes are parallel – intersect in a line.

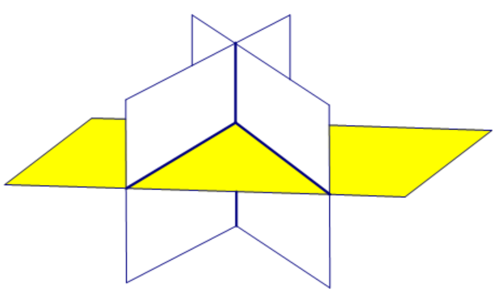


- The normals are coplanar but not parallel

➢ All three planes intersect at one line
➢ Infinite solutions in the form of a line

8. None of the three planes are parallel – intersect in a point.

- The normals are not parallel and not coplanar



➢ All three planes intersect at one point
One distinct solution in the form of a point

Definitions:
A system of three planes is **consistent** if it has one or more solutions.
A system of three planes is **inconsistent** if it has no solution.

Method for solving (possible) intersection of 3 planes

1. Check the normals for parallel or coincident planes. Suppose three distinct planes have normal vectors $\vec{n}_1$, $\vec{n}_2$, $\vec{n}_3$. To determine if there is a unique point of intersection, calculate $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3)$
   - If $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3) \neq 0$, the normal vectors are not coplanar.
     - **There is a single point of intersection**
   - If $\vec{n}_1 \bullet (\vec{n}_2 \times \vec{n}_3) = 0$, the normal vectors are coplanar.
     - **There may or may not be points of intersection**
       - If there are any points of intersection then they lie on a line

2. Using Π_1 and Π_2, solve for 2 variables in terms of the third (similar to intersection of 2 planes).
3. Substitute new equations into Π_3 and interpret results:
   a) Point (solution of parameter eg. $t = 4$)
   b) Line (infinite possible # of solutions for parameter eg. $0t = 0$)
   c) Plane (infinite possible # of solutions eg. $\Pi_1 = k\Pi_2 = m\Pi_3$)
   d) None (if $0t = \#$)

Introduction to Matrices

Mathematicians often develop new notation and ideas to help ease complicated calculations or procedures. The process of solving the system can be simplified by using a matrix to help organize and eliminate variables efficiently.
Informally, a **matrix** (the plural form is matrices) is an array of *m* rows × *n* columns.
For example,

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & -4 & 5 \\ 1 & 2 & 7 \end{bmatrix}$$

A is a 3×3 matrix since it has 3 rows and 3 columns.
We often use a single, capital letter to represent a matrix, such as *A* in our example.
Further, A_{ij} is the notation used to reference the element in the i^{th} row and j^{th} column of matrix *A*. In this example, $A_{31} = 1$.

ROW MATRIX: $\begin{bmatrix} 1 & 7 & 5 & -1 \end{bmatrix}$ ***COLUMN MATRIX:*** $\begin{bmatrix} 6 \\ 0 \\ 15 \end{bmatrix}$ ***SQUARE MATRIX***: $\begin{bmatrix} 6 & -1 & 7 \\ 5 & 7 & 9 \\ -4 & -5 & 0 \end{bmatrix}$

To translate a system of linear equations into matrix form, we write the coefficients and the constant terms of the linear equations as elements in the corresponding locations in the matrix. From example

$$\begin{aligned} 1x + 2y - 4z &= 3 \\ -2x + 1y + 3z &= 4 \\ 4x - 3y - 1z &= -2 \end{aligned}$$

becomes

$$\left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

Such a matrix that includes the constant terms is known as an **augmented** matrix, and the elements in the matrix to the left of the vertical line form the **coefficient** matrix.

Gaussian Elimination

The benefit of using matrices becomes obvious when solving linear systems using a process known as Gaussian elimination.

In Gaussian elimination, the aim is to transform a system of equations in augmented matrix form, such as

$$A = \left[\begin{array}{rrr|r} 1 & 2 & -4 & 3 \\ -2 & 1 & 3 & 4 \\ 4 & -3 & -1 & -2 \end{array}\right]$$

into another augmented matrix of the form

$$D = \left[\begin{array}{ccc|c} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{array}\right]$$

where *D* is known as a matrix in row **echelon form**.

Row Operations

To transform the original augmented matrix into row echelon form, we perform row operations.

Possible row operations are as follows:

- Multiplying each entry in one row by a (non-zero) scalar
- Adding (or subtracting) one row to (from) another
- A combination of the above two row operations
- Interchanging rows

Examples: Investigate the intersection of the planes.

$\pi_1 : 2x - y + 3z - 2 = 0$
$\pi_2 : 4x - 2y + 6z - 3 = 0$
$\pi_3 : x - 3y + 2z + 10 = 0$

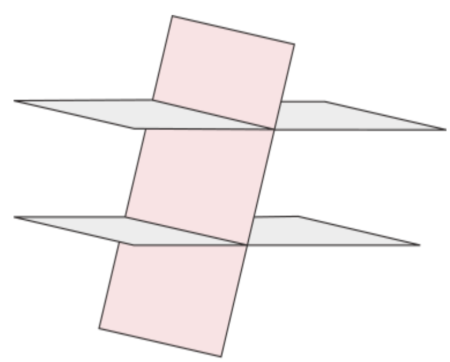
$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 4 & -2 & 6 & 3 \\ 1 & -3 & 2 & -10 \end{array}\right] R_1 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 4 & -2 & 6 & 3 \\ 2 & -1 & 3 & 2 \end{array}\right] (-2)\times R_3 + R_2 \rightarrow R_3 \left[\begin{array}{ccc|c} 1 & -3 & 2 & -10 \\ 4 & -2 & 6 & 3 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The third row of the matrix corresponds to the equation $0y = -1$, which has no solution. Thus, this system of equations has no solution and therefore, the three corresponding planes have no points of intersection.

To visualize this geometrically, we note that

$$\vec{n_2} = 2\vec{n_1}$$
$$[4,-2,6] = 2[2,-1,3]$$
$$D_2 \neq 2D_1$$

The normal vector of the third plane, $\vec{n_3} = [1,-3,2]$ is not parallel to either of these, so the third plane must intersect each of the other two planes in a line.



Using your GDC:

1. Press [2nd] [x⁻¹] for the "matrix" menu. Cursor to the right to the EDIT menu. Choose the number for the matrix you want to edit. Enter the number of rows, and the number of columns you want for the matrix when prompted for [ ]x[ ], then press [enter]. For example 3x4 for a system of 3 equations.
2. Enter the values of the coefficients and the constant from the system into the matrix. When you are finished entering the values, press [2nd] [mode] to go back to the home screen.
3. Press [2nd] [x⁻¹] for the "matrix" menu, cursor to the right to the MATH menu. Scroll down and choose B: rref(for "reduced row echelon form" then press enter.
4. When prompted for the matrix name, press [2nd] [x⁻¹] for the "matrix" menu, then choose the number for the matrix you are working with under the NAMES menu. Press enter.
5. The screen should give you the solution in the form of $\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ where the solution is $x = a, y = b, z = c$.

Investigate the intersection of the following planes.

1) $$\begin{aligned} \pi_1 &: 2x - y + 3z - 2 = 0 \\ \pi_2 &: x - 3y + 2z + 10 = 0 \\ \pi_3 &: 5x - 5y + 8z + 6 = 0 \end{aligned}$$

2) $\left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & -3 & 2 & -10 \\ 5 & -5 & 8 & -6 \end{array}\right]$

3)
$$\pi_1 : x + 3y - z + 9 = 0$$
$$\pi_2 : x - y + z - 11 = 0$$
$$\pi_3 : x + 2y + 4z - 5 = 0$$

4) $$\begin{cases} \pi_1 : 2x + 3y + z - 1 = 0 \\ \pi_2 : x - y + z - 2 = 0 \\ \pi_3 : 2x - 2y + 2z - 4 = 0 \end{cases}$$

5) $\begin{cases} 2x + y - 3z + 5 = 0 \\ x + y + z - 6 = 0 \\ 4x - 5y + z + 3 = 0 \end{cases}$

Practice

1. Given the related augmented matrix, find the solution to the system

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

c. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 12 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 4 \end{array}\right]$

b. $\left[\begin{array}{ccc|c} 0 & 2 & 0 & 6 \\ 0 & 0 & -3 & 9 \\ 4 & 0 & 0 & 8 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 7 \\ 3 & 1 & 0 & 4 \\ 0 & 0 & 0 & -5 \end{array}\right]$

2. Use Gauss-Jordan elimination to find the intersection of each of the following sets of planes

a. $\begin{aligned} x+3y-z+9&=0 \\ x-y+z-11&=0 \\ x+2y+4z-5&=0 \end{aligned}$

b. $\begin{aligned} x-y-2z-5&=0 \\ 2x+2y+z-1&=0 \\ x+3y+3z-10&=0 \end{aligned}$

c. $\begin{aligned} 2x+3y+z-1&=0 \\ x-y+z-2&=0 \\ 2x-2y+2z-4&=0 \end{aligned}$

d. $\begin{aligned} 2x-y+3z&=0 \\ -2x-3y+2z+10&=0 \\ 5x-5y+8z+6&=0 \end{aligned}$

3. Find value(s) of a and b so that the planes

$$\begin{aligned} 2x+y-z&=0 \\ x+2y+3z&=0 \\ 3x+ay+2z+b&=0 \end{aligned}$$

(a) intersect in a line
(b) intersect at a point.

4. For what value of k will the following set of planes interesting in a line?

$\pi_1 : x - 2y - 3z = 0$

$\pi_2 : x + 9y - 5z = 0$

$\pi_3 : kx - y + z = 0$

Answers

1. a. $(3,2,-5)$
   b. $(2,3,-3)$
   c. $(-4,-1,4)$
   d. No solution.

2. a. $(5,-4,2)$
   b. No solution.
   c. $\vec{r}=[0,2,0]+t[2,1,-1]$
   d. $\vec{r}=[\frac{16}{5},\frac{22}{5},0]+t[7,-1,5]$

3. a. $a=3, b=0$
   b. $a\neq 3, b\in R$

4. $k=\frac{-9}{37}$

Warm -up

Determine the intersection of the following planes

a. $\begin{aligned} x - y + z &= -2 \\ 2x - y - 2z &= -9 \\ 3x + y - z &= -2 \end{aligned}$ Ans: $[(-1,3, 2)]$

b. $\begin{aligned} x + y + 2z &= -2 \\ 3x - y + 14z &= 6 \\ x + 2y &= -2 \end{aligned}$ Ans: $[x, y, z] = [1, -3, 0] + [-4, 2, 1]$

c. $\begin{aligned} 3x + 2y - z &= 0 \\ 3x - 5y + 4z &= 3 \\ 2x - y + z &= 1 \end{aligned}$ Ans: $\vec{r} = [\frac{2}{7}, -\frac{3}{7}, 0] + t[-1, 5, 7]$

d. $\begin{aligned} x - y + 4z &= 5 \\ 3x + y + z &= -2 \\ 5x - y + 9z &= 1 \end{aligned}$ Ans: No intersection exists

1. Find the distance between the following skew lines

a) $L_1 : [x, y, z] = [-2, 1, 1] + t[3, 1, 1]; t \in R$
$L_2 : [x, y, z] = [7, -4, 3] + s[-1, 1, 0]; s \in R$

b) $\ell_1 : \frac{x}{3} = \frac{y-4}{4} = z - 8$
$\ell_2 : \frac{x+1}{2} = \frac{y-5}{5} = \frac{z+2}{2}$

2. **Solve** the following systems of equations. Give a **geometrical interpretation** of the system and its solution.

a) $\pi_1 : x - 5y + 2z - 10 = 0$
$\pi_2 : x + 7y - 2z + 6 = 0$
$\pi_3 : 8x + 5y + z - 20 = 0$

b) $\pi_1 : 2x + y + 6z - 7 = 0$
$\pi_2 : 3x + 4y + 3z + 8 = 0$
$\pi_3 : x - 2y - 4z - 9 = 0$

3. Determine the shortest **exact** distance from the point $A(1, 2, 3)$ and the line $\begin{cases} x = 3s \\ y = 1 + 4s \\ z = 5 + s \end{cases}$.

4. The shortest distance between the point (1,3,2) and the line $\frac{x-1}{-1} = \frac{y-3}{1} = \frac{z-k}{2}$ is $\sqrt{3}$ units. Find the value(s) of k.

5. Consider the following three planes:
① $4x + 3y + 3z = -8$
② $2x + y + z = -4$
③ $3x - 2y + (m^2 - 6)z = m - 4$

Determine the value(s) of m for which the planes intersect in a point.

6. Prove that the following planes may intersect in a line provided that $a^2 + b^2 + c^2 = 1 - 2abc$.

$\pi_1 : -x + ay + bz = 0$
$\pi_2 : ax - y + cz = 0$
$\pi_3 : bx + cy - z = 0$

Quiz: Equations Of Planes Name: _______________

1. Determine two points on the plane $\overrightarrow{OP} = [2,0,-1] + s[0,2,-1] + t[5,3,4]$.

2. Determine the parametric equations for the plane $[x - 2, y, z-3] = s[1,1,1]+t[0,3,5]$.

3. Give a vector equation for the plane described by $x = 3-s+4t$, $y=2t$, $z=1+4s-5t$.

4. Find a normal to the plane $\vec{r} = [3,1,-6] + s[0,1,1] + t[-1,2,1]$.

5. Find vector and parametric equations of each plane.

a) plane parallel to xy- plane through the point (l, 3, 5).

b) The plane through (3, -5, 1) and parallel to the plane given by

$$x = s + 3t - 5, y = 2 + s - t, z = s + t - 5$$

c) The plane through the points (1, -3, 2), (0, 1, -2) and (9, 2, 0).

6. Find a scalar equation for each plane .

a) The plane with direction vectors [1, 5, -2] ,[0,0,1] through the point (3,4,1).

b) The plane containing the line $[x, y, z] = [1, 3, -1] + r[3, 2, 2]$ and parallel to the $x = 2 + 3s, y = -2s, z = 4$

c) The plane through the point (0,2,-1) and perpendicular to the line [x, y, z] =t[3,-2,4]